The Dissertation Committee for Saša Kocić

certifies that this is the approved version of the following dissertation:

Renormalization of continuous-time dynamical systems with KAM applications

Committee:

Hans Koch, Supervisor

Rafael de la Llave

Richard Hazeltine

Michael Marder

Jack Swift

Renormalization of continuous-time dynamical systems with KAM applications

by

Saša Kocić, BS, MS

Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Austin

December 2006

Acknowledgments

I would like to express my deepest gratitude to my advisor Professor Hans Koch for his constant advice and help over the years. I am thankful to him for always being available for me, teaching me so many things, and helping me to bridge the gap between math and physics. I have also learned a great many things from other people at the University of Texas. I would like to acknowledge discussions with Professor Rafael de la Llave, which inspired my interest in the more mathematical aspects of dynamical systems. Discussions and work with Professor Richard Hazeltine on magnetohydrodynamics and Professor Michael Marder on fracture dynamics have broadened my research views and interests into dynamics-related areas of physics. I was honored to listen to the lectures on nonlinear dynamics by Professor Jack Swift. I would like to use this opportunity to thank them all for numerous discussions that I deeply enjoyed and their great help in my career. I would also like to thank everybody else from the Mathematics and Physics Departments who have helped me on this path.

I would also like to acknowledge the continuous love and support of my family and friends. In particular, I'm thankful to my parents and Ana, Ben, Biny, Vladica and Vladimir for their help. Finally, I would like to mention my grandparents who passed away in recent years. It is the warmness of their home that I still carry

deeply in my heart.

SAŠA KOCIĆ

The University of Texas at Austin

December 2006

Renormalization of continuous-time dynamical systems with KAM applications

Publication No. ______________________

Saša Kocić, Ph.D.
The University of Texas at Austin, 2006

Supervisor: Hans Koch

In this dissertation, we construct a sequence of renormalization group transformations on a space of analytic vector fields. We apply these transformations to study the persistence of quasiperiodic motion (invariant tori) with sufficiently incommensurate frequency vectors ω in near-integrable systems. The renormalization transformations preserve geometrical "classes" of the vector fields, such as Hamiltonian, divergence-free, time-reversible, and symmetric with respect to an involution. Two different approaches have been developed. One approach makes use of a recent multidimensional generalization of the continued fraction algorithm and applies to Diophantine frequency vectors ω. The other approach applies to the larger set of

Brjuno frequency vectors. We prove the existence of an integrable limit set of the renormalization and show that there exists a finite-codimension stable manifold $\mathcal{W}$ for the sequence of renormalization maps, associated to this set. We show that every vector field on $\mathcal{W}$ has an analytic elliptic invariant torus on which the flow is conjugate to a rotation with a Diophantine or, more generally, Brjuno frequency vector ω. Consequently, every family of vector fields that intersects $\mathcal{W}$ has a member which has an analytic invariant torus with frequency vector ω. We show that the number of parameters of a family can be reduced if a non-degeneracy condition is satisfied. In certain classes of vector fields, e.g. Hamiltonian vector fields, the number of parameters can be reduced to zero, and analogous statements are true for individual vector fields.

In the special case of two degree of freedom Hamiltonian vector fields we also construct a sequence of renormalization group transformations with an attracting integrable limit set, directly on a space of Hamiltonian functions. As an application of the scheme we give a proof of KAM theorem for Hamiltonians satisfying a non-degeneracy condition. On a numerical level, the scheme can be applied to obtain the critical function of one-parameter families of two-degree of freedom Hamiltonian systems.

Contents

Chapter 1

Introduction

In this dissertation, we develop a renormalization group method for continuous-time dynamical systems. We construct three renormalization schemes and apply them to study the persistence of quasiperiodic motion (invariant tori) with Diophantine and, more generally, Brjuno frequency vectors in near-integrable systems. The method itself can have broader applications and we hope will be used to study the critical behavior of systems far from integrability. In this introduction, we discuss the stability of motion in dynamical systems and give an overview of the dissertation results in the context of renormalization in dynamical systems.

1.1 Stability of motion

Loosely speaking, there are two types of systems in nature: systems that have very unstable behavior and systems that appear to be stable. An example of the former is weather and an example of the latter is the planetary motion of our Solar system. The dynamics of many systems in nature can be modeled in a mathematical setting called a dynamical system. One of the main problems in the theory of dynamical systems, is to decide which systems have stable and which have unstable behavior.

Dynamical systems that are called *integrable* have stable and well-understood dynamics. A particularly important class of systems are Hamiltonian systems (see Chapter 2 for a precise definition). A Hamiltonian system with d degrees of freedom is called integrable if it has d independent constants (integrals) of motion which are in involution. Two functions are said to be in involution if their Poisson bracket is equal to zero, i.e. if their corresponding flows commute. The dynamics of an integrable Hamiltonian system is constrained to the level sets of these integrals of motion which are d-dimensional tori. Motion on these *invariant tori* is stable and conjugate to a rotation with frequency vector $\omega \in \mathbb{R}^d$. Therefore, it can be periodic or quasi-periodic.

Since systems in nature are not isolated, an important question is whether these invariant tori persist under small perturbations. When studying systems which are close to a system with known behavior, an important mathematical tool is perturbation theory. In dynamical systems, one can attempt to apply perturbation theory to systems close to an integrable system. A naive approach to perturbation theory in dynamics, that involves the expansion of a solution of an autonomous dynamical system in powers of a perturbation parameter, often leads to a series that is unable to predict the long-time behavior of the system. This was first realized in studies of stability in celestial mechanics, a problem which dates back to Newton. The study of the Earth-Moon-Sun system by Charles Delaunay (in the second half of the nineteenth century), as a special case of the three body problem, demonstrated for the first time the problem of *small denominators* in perturbation theory [13, 14]. Modern formulation of the problem is due to Poincaré who was the first to recognize the complicated geometry of trajectories in the three body problem, and the importance of motions which we today call *chaos*.

A fundamental problem in celestial mechanics was to construct quasiperiodic solutions in the N body problem. This problem led to the introduction of a formal quasiperiodic perturbation series. More specifically, given a frequency vector ω, one can find a series in the perturbation parameter whose coefficients are analytic quasiperiodic functions with frequency vector ω. Poincaré undertook the studies of the convergence of these series. He named it after Lindstedt who had himself made a significant contribution to the solution of the problem. Poincaré constructed the general quasiperiodic solution with varying frequency, and showed that it is not uniformly convergent in the initial conditions and the perturbation parameter. Poincaré did not solve the problem of convergence of the Lindstedt series for a fixed frequency (corresponding to fixed initial conditions), though he seemed to have favored the divergence [70].

In 1954, Kolmogorov stated a theorem, which was later proved and extended by Arnol'd and Moser. The theorem guarantees the existence of analytic quasiperiodic solutions with Diophantine frequency vectors in Hamiltonian systems sufficiently close to a non-degenerate integrable system. Due to the analyticity of the solutions, the terms in their expansion have to be those given by the Lindstedt series. Thus, in an indirect way, KAM (Kolmogorov-Arnol'd-Moser) theory shows that the series do in fact converge for a fixed frequency. The convergence of Lindstedt series in a direct way was shown by Eliasson [20].

KAM theory is not just a collection of theorems, but rather a collection of methods and tools for dealing with small denominators [55]. It started with the seminal work of Kolmogorov [51], and was basically shaped by Arnol'd [3] and Moser [66], in 1960s, but the development never stopped [9, 10, 67, 71–73]. Some particular work includes extensions from Diophantine to a larger set of frequency vectors [5, 7, 8, 37] and a recent approach to KAM theory inspired by quantum field theory and Feynman diagrams [6, 28–31]. It is an attempt of this dissertation to further contribute to this development. The renormalization schemes constructed in this dissertation provide a method for dealing with small denominators and a tool for the construction of invariant tori with Diophantine and, more generally, Brjuno frequency vectors.

1.2 Renormalization in dynamical systems and overview of the results

The idea of renormalization originated in theoretical physics. It was invented first in quantum field theory by Stueckelberg and Peterman [76] in the 1950's. Renormalization group ideas were later introduced in statistical mechanics by Kadanoff [38], and successfully applied to the study of critical phenomena. In the theory of dynamical systems, renormalization group techniques were first introduced by Feigenbaum [23–25] and Coullet and Tresser [17], in the late 1970's, in order to explain the apparent universality of period-doubling sequences in one-parameter families of one-dimensional maps. Since then, renormalization group ideas have been widely applied in the investigations of a variety of dynamical phenomena [35,39,54,61–65,68,74,75].

In the context of Hamiltonian systems, renormalization group ideas were first introduced by Escande and Doveil [22] in the early 1980's. They considered the problem of the break-up of invariant tori in two degree of freedom Hamiltonian flows. Almost at the same time, renormalization ideas were applied to two-dimensional area-preserving maps by MacKay [61]. This type of map naturally appears in the Poincaré sections of the flows of two degree of freedom Hamiltonian systems. MacKay considered the problem of constructing invariant curves with golden mean rotation number for two-dimensional maps of the cylinder. Later a different scheme was applied by Khanin and Sinai [40] for more general Diophantine rotation numbers. Since a cylinder map can be lifted to a map on a plane that commutes with certain group of translations, the renormalization transformations in these approaches were essentially constructed on a space of pairs of commuting maps. The commutativity condition, however, led to a technical problem that can be avoided if the renormalization transformations are constructed directly on a space of Hamiltonian functions or on a space of vector fields (as has been done in this dissertation).

In the Hamiltonian context, this idea was first realized by Koch [44] in the late 1990's. He constructed a rigorous renormalization scheme for analytic Hamiltonian functions (referred to in the following as Hamiltonians). The scheme is applicable to Hamiltonians close to an integrable Hamiltonian which has an invariant torus of a given frequency vector. The scheme applies to Hamiltonians with an arbitrary number of degrees of freedom, but the set of allowed frequency vectors has zero Lebesgue measure (moreover it is countable). For the subsequent related work, the interested reader is directed to references [1, 2, 12, 15, 16, 21, 26, 27, 33, 48, 56].

A part of this dissertation (Chapter 3) is the formulation of a renormalization scheme for analytic Hamiltonians that applies to a set of full Lebesgue measure Diophantine frequency vectors [50]. The scheme applies to two degree of freedom Hamiltonians. In this case, the corresponding frequency vectors are two-dimensional, and the motion on invariant tori is essentially characterized by single numbers, i.e. the ratios of their components. Assuming that these numbers are irrational, the renormalization transformations are related to the sequence of their rational approximates, generated via the one-dimensional continued fraction algorithm. This algorithm was also used for the construction of a renormalization scheme for flows on a torus of dimension two by Lopes-Dias [57] (see [58–60] for

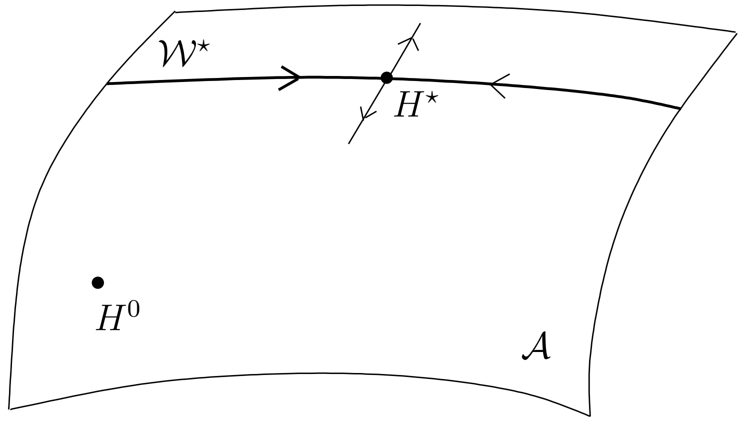

Figure 1.1: *Trivial and nontrivial fixed points H^0 and $H^\star$ of a renormalization operator in a Banach space $\mathcal{A}$ of two degree of freedom Hamiltonians.*

related work).

As an application of our scheme, we give a proof of KAM theorem for near-integrable Hamiltonians [50]. In the renormalization language, KAM theory corresponds to a neighborhood of the integrable (trivial) limit set of the renormalization transformations. Numerical experiments suggest the existence of another (nontrivial) limit set in the space of the two degree of freedom Hamiltonians. The existence of such a nontrivial limit set, more precisely of a nontrivial fixed point, had been conjectured earlier for the renormalization of area preserving maps. For the renormalization of Hamiltonians, a computer-assisted proof was given by Koch [46] that, in the case of golden mean rotation number (frequency ratio), there exists a non-trivial fixed point of the renormalization transformations. This fixed point corresponds to a (non-differentiable) invariant torus at the break-up [47] (Figure 1.1).

To extend the construction of the renormalization scheme to higher degree of freedom Hamiltonians, one may attempt to use an appropriate generalization of the one-dimensional continued fraction algorithm. Although many multidimensional continued fraction algorithms have been known for a long time, such as the Jacobi-Perron algorithm [4], they do not have all of the nice properties of the one-dimensional algorithm and are not well-suited for applications in dynamics. In particular, it is not known whether these algorithms provide simultaneous rational approximations for Lebesgue almost-all vectors, and for some algorithms it is known that there exist vectors for which the approximations do not converge. Recently, an algorithm was introduced by Khanin, Lopes-Dias and Marklof [41], based on work concerning flows on homogeneous spaces by Dani [18], Lagarias [53] and Kleinbock and Margulis [43], that is appropriate for these applications. The authors used the algorithm to extend the renormalization scheme for Hamiltonians for Diophantine frequency vectors to arbitrary dimensions [41, 42]. Attempts to extend the renormalization scheme for Hamiltonians to a larger set of frequency vectors have not been successful so far.

In this dissertation we construct a renormalization scheme for general analytic vector fields. We develop two approaches to the construction of the renormalization operator. In one, which applies to Diophantine frequency vectors, we make use of the above multidimensional continued fraction algorithm (Chapter 4). In another approach, which does not use a continued fraction algorithm, we were able to construct a renormalization scheme that applies to the larger set of Brjuno frequency vectors (Chapter 5).

The essential property of the scheme is that many subsets (Lie algebras) of vector fields, are left invariant under the renormalization transformations. In the following we will refer to these subsets as *classes*. In particular, renormalization preserves the classes of Hamiltonian or divergence-free vector fields, and the classes of vector fields which are symmetric or reversible with respect to an involution. We prove the existence of an integrable limit set of the renormalization, and show that there exists a stable manifold $\mathcal{W}$ for the sequence of renormalization transformations, which is of finite codimension. As an application of this renormalization scheme, we prove that every vector field on $\mathcal{W}$ has an invariant torus on which the motion is conjugate to a rotation with a given Diophantine or, more generally, Brjuno frequency vector ω. Consequently, we construct the analytic invariant tori with these frequency vectors in near-integrable families of vector fields, that intersect $\mathcal{W}$. The number of parameters in the families of vector fields needed to assure the existence of invariant tori depends on the particular class under consideration. Furthermore, this number can be reduced, if we assume that a non-degeneracy condition is satisfied. We determine the minimal number of parameters in some classes of vector fields.

In the next chapter we provide some background in continuous-time dynamical systems and KAM theory in the context of Hamiltonian flows. Later chapters contain constructions and applications of the renormalization schemes discussed above. Though the applications we present here concern only the construction of invariant tori in systems which are close to integrable, it should be noted that the basic tools we have constructed, i.e. the renormalization schemes, can be applied to systems far from integrability. We hope that the methods presented here will lead to a better understanding of systems in that regime and expose non-trivial limit sets of the renormalization transformations. The only rigorous result in this area so far is the above-mentioned proof of existence of the non-trivial fixed point of the renormalization operator in the case of two degree of freedom Hamiltonians and the golden mean frequency ratio. Many problems are still open, even on a numerical level.

Chapter 2

Background

2.1 Continuous-time dynamical systems

A dynamical system is a mathematical setting with a prescribed rule for the time evolution. It is usually a manifold M, called the phase (or state) space, on which the dynamical law is given by a map $f : \mathbb{N}_0 \times M \to M$, or by a system of ordinary differential equations,

$$\frac{dz}{dt} = X(z,t), \tag{2.1.1}$$

with $z \in M$. In the first case, the dynamical law leads to a discrete evolution of the system initially at state $z_0 \in M$, through the sequence of points defined recursively as $z_{n+1} = f(n, z_n)$, for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In the second case, an initial state $z_0 \in M$ evolves continuously with the flow Φ_X of the vector field X, i.e $z(t) = \Phi_X^t(z_0)$, $t \in \mathbb{R}$. The former and latter are referred to as discrete and continuous-time dynamical systems, respectively. If the dynamical law does not depend explicitly on time t in the continuous or n in the discrete case, the dynamical system is called autonomous. An autonomous dynamical system is thus given by a map $f : M \to M$ or by a vector field $X : M \to M$. To a non-autonomous dynamical system, can be associated an autonomous one on the extended phase space obtained by adding an additional independent variable. For this reason, the study of dynamical systems is often focused on autonomous ones. The one parameter family of diffeomorphisms generated by the flow of an autonomous system satisfies the group (composition) property $\Phi_X^{t+s} = \Phi_X^t \circ \Phi_X^s$. In a wider sense of the word, a dynamical system is understood to mean an arbitrary action of a group (or even of a semi-group) on a certain set which is then named the phase space.

We will only consider autonomous continuous-time dynamical systems. We will also assume that the vector fields X are analytic. For analytic vector fields one has, among other things, Cauchy's theorem (further extended by Kovalevskaya in her dissertation [52]) which guarantees the existence of a unique local solution of the system of equations (2.1.1) through a given point. Therefore, every point on M has a unique future and past evolution.

In the following, we describe some important classes of vector fields X and the corresponding dynamical systems, arising from their geometrical structures.

(i) Volume-preserving vector fields: A vector field X on an n-dimensional manifold M, $n \in \mathbb{N}$, is called volume preserving if there exists a non-degenerate n-form (called volume form) which is preserved by the flow of the vector field, i.e. $(\Phi_X^t)^*\omega = \omega$. Here, asterisk denotes the pullback under the map Φ_X^t. We will consider only volume-preserving vector fields that preserve the form

$$\Omega = dz_1 \wedge \cdots \wedge dz_n, \tag{2.1.2}$$

where $z = (z_1, \ldots, z_n)$.

(ii) Hamiltonian vector fields: A vector field X on an even-dimensional manifold M is called a Hamiltonian vector field, if it can be derived from a function H on M, called the Hamiltonian function, in the following way. If M is a $2d$-dimensional manifold and $(q_1, \dots, q_d, p_1, \dots, p_d)$ is a set of coordinates on M, then $X = (\nabla_p H, -\nabla_q H)$. Such a vector field corresponds to a dynamical system in the canonical form of Hamilton's equations

$$\dot{q} = \nabla_p H\,, \qquad \dot{p} = -\nabla_q H. \tag{2.1.3}$$

Here, and in what follows, dot denotes the derivative with respect to time.

(iii) Vector fields with an involutive symmetry: A vector field X on M is symmetric with respect to an involution if there is a map $G : M \to M$ which is an involution, i.e. $G \circ G = \mathrm{I}$, and

$$(DG)^{-1} X \circ G = X. \tag{2.1.4}$$

Here, and in the following, D stands for the derivative of a map.

(iv) Reversible vector fields: A vector field X on M is called time-reversible with respect to an involution if there exists an involution $G : M \to M$ such that

$$(DG)^{-1} X \circ G = -X. \tag{2.1.5}$$

(v) Vector fields that generate a skew-product flow: A vector field X of the form $X(x, y) = (\omega, A(x)y)$, where $A : \mathbb{T}^d \to \mathbb{R}^\ell \times \mathbb{R}^\ell$, for $(x, y) \in \mathbb{T}^d \times \mathbb{R}^\ell$, will be called here a vector field with skew-product flow. The flow on the base space $\mathbb{T}^d$ is linear and given by $x \mapsto x + \omega t$, $t \in \mathbb{R}$. On the fiber space $\mathbb{R}^\ell$, the dynamics is given by the equation $\dot{y} = A(x)y$.

(vi) Vector fields on a torus: The vector field of the form $Z(x, y) = (f(x), 0)$, where f is a map from $\mathbb{T}^d$ to $\mathbb{R}^d$ and $0 \in \mathbb{R}^\ell$, is called a vector field on a torus. Its flow leaves invariant every d-dimensional torus labeled by the constant $y \in \mathbb{R}^\ell$.

2.2 Hamiltonian systems

A precise definition of a Hamiltonian system requires an even-dimensional manifold M equipped with a symplectic structure. Such a structure on M is given by a closed non-degenerate 2-form ω^2, called the symplectic form. The condition that the form is closed means that $d\omega^2 = 0$ and the condition that it is non-degenerate means that if $\omega^2(X, Y) = 0$ for every vector field Y on M, then $X = 0$. The pair (M, ω^2) is called the symplectic manifold.

The symplectic form provides a natural correspondence between vector fields and 1-forms, since, due to non-degeneracy of the form, the map $X \mapsto i_X\omega^2 := \omega^2(X, \,\cdot\,)$, from the tangent into the cotangent bundle, provides an isomorphism between the tangent and cotangent space at every point on M.

A vector field X on a $2d$-dimensional manifold M is called symplectic if its flow preserves the symplectic form, i.e. if $\Phi_X^*(\omega^2) = \omega^2$. This implies that the Lie derivative $L_X\omega^2 = 0$, and, since ω^2 is closed, using Cartan's formula for the Lie derivative of a form ϑ,

$$L_X\vartheta = d(i_X\vartheta) + i_X(d\vartheta), \tag{2.2.1}$$

we find the $di_X\omega^2 = 0$, i.e. that the 1-form associated to X is closed.

If the 1-form $i_X\omega^2$ is exact, i.e. if there is a function H such that $i_X\omega^2 = -dH$, the vector field is called Hamiltonian. Symplectic and Hamiltonian vector fields are also referred to as locally and globally Hamiltonian vector fields, since by the Poincaré lemma every closed form is locally exact.

Assuming that $(q,p) := (q_1,\ldots,q_d,p_1,\ldots,p_d)$ are coordinates on the manifold M, a fundamental example of a symplectic form is

$$\omega^2 = \sum_{i=1}^{d} dp_i \wedge dq_i. \tag{2.2.2}$$

With this symplectic form, the identity $i_X\,\omega^2 = -dH$, gives the standard form of the Hamiltonian vector field and Hamilton's equations (2.1.3). The existence of coordinates in which an arbitrary symplectic form takes this form is guaranteed locally by Darboux's theorem.

The symplectic form ω^2 induces a volume form Ω on M,

$$\Omega = \underbrace{\omega^2 \wedge \cdots \wedge \omega^2}_{d\ \text{times}}. \tag{2.2.3}$$

Obviously, the flow of a vector field that preserves the symplectic form also preserves Ω. This fact is known as Liouville's theorem.

Of great importance are the changes of coordinates that preserve the symplectic form. If U is such a diffeomorphism, i.e. $U^*\omega^2 = \omega^2$ and if X is a Hamiltonian vector field generated by the Hamiltonian H, i.e. $i_X\,\omega^2 = -dH$, then

$$i_{U^*X}\,\omega^2 = i_{U^*X}\,U^*\omega^2 = U^*i_X\,\omega^2 = -U^*dH = -d(U^*H) = -d(H\circ U). \tag{2.2.4}$$

Thus, the transformed vector field U^*X is also Hamiltonian with the Hamiltonian $H\circ U$. In the physics literature, symplectic diffeomorphisms are usually called canonical transformations, and this property is described as: canonical transformations preserve Hamilton's equations.

On the space of functions on M, we can define the Poisson bracket in the following way. For any two functions f and g, let X_f and X_g, be the associated Hamiltonian vector fields, i.e. $i_{X_f}\,\omega^2 = -df$ and $i_{X_g}\,\omega^2 = -dg$. The Poisson bracket of f and g is defined as

$$\{f, g\} := -\omega^2(X_f, X_g). \tag{2.2.5}$$

Clearly, the Poisson bracket is bilinear and antisymmetric.

In the coordinates (q, p), where the symplectic form is given by (2.2.2), the Poisson bracket of functions f and g takes the form

$$\{f, g\} = \nabla_q f \cdot \nabla_p g - \nabla_q g \cdot \nabla_p f. \tag{2.2.6}$$

It is easy to check that the Poisson bracket satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0. \tag{2.2.7}$$

Thus, the space of functions on a symplectic manifold M with Poisson bracket is a Lie algebra. The Poisson bracket is also a derivation on the algebra of functions on M, i.e. it satisfies Leibniz's product rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}. \tag{2.2.8}$$

Therefore, the space of functions on M with a product and the Poisson bracket forms a Poisson algebra.

Using Cartan's notation, one can define the commutator of two vector fields X and Y as $[X, Y] = XY - YX$. Given two Hamiltonian functions f and g, one can show that

$$[X_f, X_g] = -X_{\{f,g\}}. \tag{2.2.9}$$

This formula establishes the relationship between the commutator of two Hamiltonian vector fields and the Poisson bracket of the corresponding Hamiltonian functions.

We note here that the use of Cartan's notation is restricted to this section.

2.3 Integrable systems

An important concept for general dynamical systems, and in particular for Hamiltonian ones, is the concept of an invariant torus. For a general vector field X on $\mathbb{T}^d \times \mathbb{R}^\ell$, an invariant d-torus, with frequency vector $\omega \in \mathbb{R}^d$, is a continuous map Γ of $D_0 = \mathbb{T}^d \times \{0\}$ into the phase space, which conjugates the flow of X to a rotation with frequency vector ω, i.e. that satisfies

$$\Phi_X^t \circ \Gamma = \Gamma \circ \Phi_K^t\,, \qquad t \in \mathbb{R}\,, \tag{2.3.1}$$

where $K = (\omega, 0)$. If the frequency vector ω of an invariant torus is commensurate (i.e. there exists a nonzero vector $\kappa \in \mathbb{Z}^d$ such that $\omega \cdot \kappa = 0$), the motion on the invariant torus is periodic. If the frequency vector ω is incommensurate (i.e. $\omega \cdot \kappa \neq 0$ for all nonzero $\kappa \in \mathbb{Z}^d$), the motion on the invariant torus is quasiperiodic.

A Hamiltonian system is called integrable if there exists a canonical change of variables, which transforms the Hamiltonian into one which depends only on the new momenta. In the new coordinates, the dynamics is then given by

$$\dot{q} = \nabla_p H(p), \qquad \dot{p} = 0, \tag{2.3.2}$$

where H is the Hamiltonian in the new coordinates. Thus, the phase space of an integrable Hamiltonian system is foliated by invariant tori, characterized by a constant value of p. These tori are Kronecker, i.e. they have the linear flow $\Phi^t : q \mapsto q + w(p)t$, where the frequency vectors are defined as $w(p) = \nabla_p H(p)$. They are also Lagrangian, i.e. the restriction of the symplectic form to their tangent space vanishes and they are maximal with respect to this property.

2.4 Perturbation theory, small denominators and KAM theory

Consider the phase space $\mathbb{T}^d \times \mathbb{R}^d$ of a Hamiltonian system. One way to generate a canonical change of coordinates $(q, p) = U(q', p')$ on the phase space is to give a function F_2 which depends on the old angles q and new momenta p'. In order to preserve the standard symplectic form, it suffices that the one-form $p' \cdot dq' - p \cdot dq$ is closed. Thus, it is sufficient that $p' \cdot dq' - p \cdot dq = d(p' \cdot q' - F_2(q, p'))$, i.e. $p = \nabla_q F_2(q, p')$, and $q' = \nabla_{p'} F_2(q, p')$. Such a function F_2 is called a type-2 generating function.

Notice that the function $F_2(q, p') = q \cdot p'$ generates the identity transformation. A map $U = \mathrm{I} + u$, close to the identity, can be generated by a function ϕ by setting $F_2(q, p') = q \cdot p' - \phi(q, p')$. We obtain that

$$u(q', p') = (\nabla_{p'}\phi(q, p'), -\nabla_q \phi(q, p'))\,. \tag{2.4.1}$$

Under this canonical transformation, the Hamiltonian transforms as $H \mapsto H \circ U$, where

$$H \circ U = H + \{H, \phi\} + \dots. \tag{2.4.2}$$

Let us consider a Hamiltonian $H = H^0 + h$, where H^0 is an integrable Hamiltonian depending only on the momenta p, and the perturbation h is of order

ε. We assume that the angle average of h is zero, up to the order of ε. Suppose that we try to perform a canonical transformation U, such that the transformed Hamiltonian $H \circ U$ depends only on the new momentum variables p'. Thus, we require that

$$H \circ U(q', p') = H^0(p') + h(q', p') - \nabla_{p'} H^0(p') \cdot \nabla_{q'} \phi(q', p') + \mathcal{O}(\varepsilon^2) \qquad (2.4.3)$$

is independent of q'.

Defining the frequencies of the unperturbed system as $w(p') = \nabla_{p'} H^0(p')$, and expanding $h(q', p') = \sum_{\nu \neq 0} h_\nu(p') e^{iq' \cdot \nu}$ and $\phi(q', p') = \sum_{\nu \neq 0} \phi_\nu(p') e^{iq' \cdot \nu}$ in Fourier series, where the sum goes over $\nu \in \mathbb{Z}^d$, we can determine ϕ by solving the equation

$$w(p') \cdot \nabla_{q'} \phi(q', p') = h(q', p'). \qquad (2.4.4)$$

Comparing the Fourier components leads to the formal solution for the generating function

$$F_2(q, p') = q \cdot p' + i \sum_{\nu \neq 0} \frac{h_\nu(p')}{\nu \cdot w(p')} e^{iq \cdot \nu}. \qquad (2.4.5)$$

Note that we have made the assumption that h_0 is of the order of ε^2. If this were not the case, it could have been achieved by a p-translation, provided that a certain non-degeneracy condition is imposed on H^0. The non-degeneracy condition usually means that the local frequency map $w : p \mapsto w(p)$ is a diffeomorphism.

The expression (2.4.5) represents a solution if the series in it converges. Clearly, for a given value of p', the formal solution for the generating function does not represent a solution for frequency vectors with commensurate components. However if the h_ν go to zero sufficiently fast as $|\nu| \to \infty$, as in the case when h is analytic, and if we assume some appropriate bounds on $w(p') \cdot \nu$, this does represent a solution for the given value of p', corresponding to an invariant torus.

Kolmogorov's original assertion was that under the non-degeneracy assumption

$$\det \left[\frac{\partial^2 H^0}{\partial p_i \partial p_j} \right] \neq 0\,, \tag{2.4.6}$$

the invariant tori whose frequency vectors are sufficiently incommensurate, in the sense that they satisfy a Diophantine condition, persist for sufficiently small perturbations. These are the vectors ω for which there exist constants $\mathcal{C} > 0$ and $\beta > 0$ such that

$$|\omega \cdot \nu| \geq \mathcal{C} |\nu|^{-(d-1+\beta)}, \tag{2.4.7}$$

for all $\nu \in \mathbb{Z}^d$. Here $|\nu| = \sum_i |\nu_i|$. For every $\beta > 0$, almost all vectors in $\mathbb{R}^d$ satisfy this condition with some $\mathcal{C} > 0$.

We remark that Hamiltonians that we will consider in this dissertation do not satisfy Kolmogorov's non-degeneracy condition. We emphasize this, since many examples of integrable systems in physics, e.g. Newtonian three body problem, are degenerate in the Kolmogorov sense. In the context of more general vector fields, we will also consider a weaker arithmetic condition on ω. It is a fundamental question in KAM theory to determine the weakest possible condition for the existence of invariant tori in near-integrable systems.

2.5 A simple example of renormalization

We will discuss here a very simple example of renormalization, that shares some of the essential features of the renormalization schemes constructed in the later sections. This has been inspired by an example due to MacKay [61].

We would like to show that all the functions, real analytic on an open neighborhood D of $0 \in \mathbb{R}$, which are of the form $F = F_0 + f$, where $F_0 = x^2/2$ and $f = \sum_{k \geq 3} a_k x^k$ with $a_k \in \mathbb{R}$, satisfy $F(0) = 0$, $F'(0) = 0$ and $F''(0) = 1$. In this sense, our claim is that, close to $x = 0$, all the graphs of all the functions of the above form look like the graph of F_0. Of course this is obvious, but we would like to come to this conclusion using a renormalization scheme.

We define a renormalization operator $\mathcal{R}$ on a space of all real analytic functions $\mathcal{A}$ on D, as

$$\mathcal{R}(F)(x) = 4F(x/2), \tag{2.5.1}$$

for every $F \in \mathcal{A}$. This renormalization operator involves a scaling of the region around $x = 0$. The function F_0 is a fixed point of this operator. In fact, the operator has a line of fixed points cF_0, with $c \in \mathbb{R}$, but others violate the condition, $F''(0) = 1$. The operator preserves all of the above properties, i.e. $\mathcal{R}(F)$ satisfies $F(0) = 0$, $F'(0) = 0$ and $F''(0) = 1$, if and only if F does. Since the operator is linear, the derivative of $\mathcal{R}$ at the fixed point F_0 is $\mathcal{R}$. The eigenvectors of this operator are the functions $f(x) = 1, f(x) = x, f(x) = x^2, f(x) = x^3, \ldots$ and the corresponding eigenvalues are $4, 2, 1, 1/2, \ldots$.

The expanding eigenvectors of the renormalization operator correspond to the functions $f(x) = 1$ and $f(x) = x$, for which the functions $F = F_0 + f$ do not satisfy all of the above properties. The neutral direction corresponds to the line of fixed points of this operator. Finally, the attracting directions, spanned by $f(x) = x^3, f(x) = x^4, \ldots$, correspond to the functions $F = F_0 + f$, that do satisfy the above properties.

We refer to the set of functions F, whose renormalization images $F_n = \mathcal{R}^n(F)$, $n \in \mathbb{N}$, approach F_0, as $n \to \infty$, as the stable manifold $\mathcal{W}$ of the renormalization. We would like to show that every $F \in \mathcal{W}$, satisfies the conditions $F'(0) = 0$, $F'(0) = 0$ and $F''(0) = 1$. This follows from the fact that the sequence $\{F_n\}$ converges uniformly to F_0 on D, all of the F_n are analytic and the sequences $\{F_n'\}$ and $\{F_n''\}$ converge uniformly to $F_0'(x) = x$ and $F_0''(x) = 1$, respectively.

The renormalization scheme for two-degree of freedom Hamiltonians constructed in this dissertation shares some similar features with this trivial example, in the case of the golden-mean frequency ratio. An operator is defined on a space of analytic functions, which has a fixed point in that space with corresponding stable and unstable manifolds. The operator preserves some properties of the considered functions, and from the approach to the fixed point one can show that all of the functions on the stable manifold share the same properties as the fixed point. For other frequency vectors, our renormalization operator may have more complicated limit sets, and transform the desired properties in a known way. Some distinguishing characteristics of the renormalization operators in this dissertation are that they are nonlinear and involve more than the simple scalings in this example.

2.6 Quadratic perturbations and momentum scaling

Consider perturbations of an integrable Hamiltonian H^0 depending on the p variables only. Assume that the motion on the invariant torus located at $p = 0$ is conjugate to a rotation with frequency vector ω, i.e. that $\omega = \nabla_p H^0(p)|_{p=0}$. We consider Hamiltonians $H = H^0 + h$, where the perturbation h is assumed to be an analytic function on a neighborhood of $\mathbb{T}^d \times \{0\}$, with $0 \in \mathbb{R}^d$, and thus can be expanded in a Taylor series in the p variables.

Clearly, this invariant torus persists under perturbations h that are at least quadratic in p, i.e. that satisfy $h(q, 0) = 0$ and $(\nabla_p h)(q, 0) = 0$, since the equations of motion are

$$\dot{q} = \nabla_p H^0(p) + \nabla_p h(q,p), \qquad \dot{p} = -\nabla_q h(q,p). \tag{2.6.1}$$

Writing $\mathcal{O}(p)$ for any terms of the order of p, we can write these equations as

$$\dot{q} = \nabla_p H^0(p) + \mathcal{O}(p), \qquad \dot{p} = \mathcal{O}(p^2). \tag{2.6.2}$$

Again, we will use a trivial example to illustrate the renormalization approach. Consider the momentum scaling map $\mathcal{S}_\mu(q,p) = (q, \mu p)$, with $\mu \in \mathbb{R}$, and a renormalization operator

$$\mathcal{R}(H) = \mu^{-1} H \circ \mathcal{S}_\mu, \tag{2.6.3}$$

on a Banach space of Hamiltonians. This renormalization operator is induced by a transformation homotopic to the identity and thus if H has an invariant torus with frequency vector ω, so does $\mathcal{R}(H)$. Moreover, we can see from the form of the operator that if the invariant torus of H is located at $p = 0$, so is the torus of $\mathcal{R}(H)$. The Hamiltonian $H^0 = \omega \cdot p$ is a fixed point of this operator. If we take $\mu < 1$, the Hamiltonians of the form $H = H^0 + \mathcal{O}(p^2)$, approach this fixed point under the renormalization and lie on the stable manifold $\mathcal{W}$ of this fixed point. From the uniform convergence of the orbit of a Hamiltonian $H \in \mathcal{W}$ to H^0, and the above stated properties of the renormalization operator, we can conclude that every Hamiltonian $H \in \mathcal{W}$ has an invariant torus of frequency vector ω located at $p = 0$.

2.7 A model family of Hamiltonians

Let us consider a model one-parameter family of time-dependent Hamiltonians,

$$H_\varepsilon(x, p_x, t) = \frac{p_x^2}{2} + \varepsilon V(x,t)\,, \tag{2.7.1}$$

with the potential $V(x,t)$ which is 2π-periodic both in x and t. In the special case of $V(x,t) = \cos(x) + \cos(x-t)$, one obtains the well-know family of Hamitonians first considered by Escande and Doveil [22].

The Hamiltonian H_0 has an invariant circle of frequency α^{-1}, where $\alpha > 1$, located at $p_x = \alpha^{-1}$. We can perform a momentum translation $p_x \mapsto p'_x = p_x - \alpha^{-1}$, to place this circle at $p'_x = 0$. Under the canonical change of coordinates $(x, p_x) \mapsto (x, p'_x)$, the Hamiltonians H_ε transform as $H_\varepsilon(x, p_x, t) \mapsto H_\varepsilon(x, p'_x, t)$, where

$$H_\varepsilon(x, p'_x, t) = \frac{p'^2_x}{2} + p'_x\alpha^{-1} + \frac{\alpha^{-2}}{2} + \varepsilon V(x,t) \tag{2.7.2}$$

This new family has an invariant circle of frequency α^{-1} located at $p'_x = 0$.

Time-dependent Hamiltonians can be mapped into time-independent ones by introducing a new spatial coordinate $q_2 = t$. We can also rename the coordinates by introducing $q_1 = x$ and $p_1 = p'_x$. In order to reproduce the same system of dynamical equations that follow from the Hamiltonian family (2.7.2), and the equation $\dot{q}_2 = 1$ that follows from the definition of q_2, we associate to the family of time-dependent Hamiltonians (2.7.2), the time-independent one-parameter family

$$\widetilde{H}_\varepsilon(q_1, q_2, p_1, p_2) = \frac{p_1^2}{2} + p_1\alpha^{-1} + p_2 + \varepsilon V(q_1, q_2)\,. \tag{2.7.3}$$

Since the H_ε are periodic both in x and t, the Hamiltonians $\widetilde{H}_\varepsilon$ are periodic in both q_1 and q_2. Here p_2 is the momentum coordinate conjugate to the spatial coordinate q_2. The constant term $\alpha^{-2}/2$ from the Hamiltonian family (2.7.2) has been dropped, since this transformation does not influence the equations of motion.

We can now define $q = (q_1, q_2)$ and $p = (p_1, p_2)$, and introduce the vectors $\omega = \ell(1, \alpha)$, with $\ell = \alpha^{-1}$, and $\Omega = (1, 0)$. In this notation, the Hamiltonians $\widetilde{H}_\varepsilon$

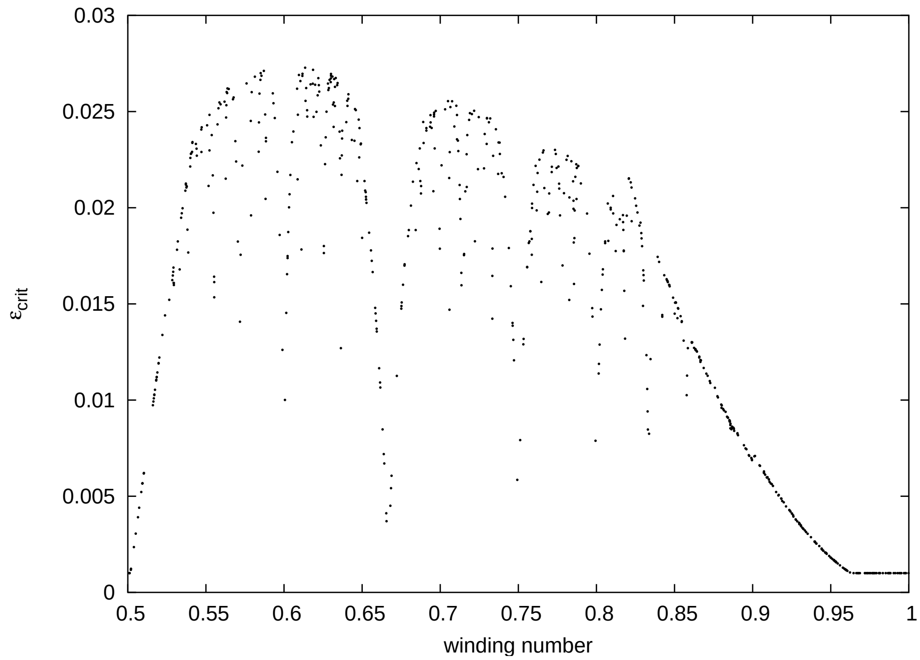

Figure 2.1: *The graph of the critical function for Escande's Hamiltonian.*

take the form

$$\widetilde{H}_\varepsilon(q,p) = \omega \cdot p + \frac{(\Omega \cdot p)^2}{2} + \varepsilon V(q)\,. \tag{2.7.4}$$

The Hamiltonian $\widetilde{H}_0$ has an invariant two-torus with frequency vector ω (and frequency ratio α), located at $p = 0$, and frequencies of nearby p-tori $\omega + \Omega \cdot p$ twisted in the direction of Ω.

In the next chapter we construct a renormalization scheme on a space of two degree of freedom analytic Hamiltonians close to an integrable Hamiltonian of the form $\widetilde{H}_0 = \omega \cdot p + \frac{1}{2}(\Omega \cdot p)^2$, allowing for perturbations that may depend on momentum coordinates p in addition to angles. The renormalization scheme is constructed for a set of full measure Diophantine frequency vectors ω. We apply the scheme to the problem of persistence of this invariant torus for small perturbations. On a numerical level, the scheme can be implemented to obtain the *critical function* for families of Hamiltonians. Assuming that the domain of attraction of the trivial limit set corresponds to the domain of existence of an invariant torus, the graph of the critical function for a one-parameter family shows the value of the parameter for which an invariant torus with a given winding number (frequency ratio) breaks. For Escande's Hamiltonian, the graph is shown in Figure 2.1.

Chapter 3

Renormalization of Hamiltonians

In this chapter we construct a renormalization scheme for two degree of freedom Hamiltonian functions. The scheme is associated to Diophantine frequency vectors which form a set of full Lebesgue measure. As an application of the scheme we give a proof of a KAM theorem for Hamiltonians degenerate in the Kolmogorov sense. Though the results of the later chapters are more general and include these results, the scheme that we will describe here applies to Hamiltonian functions, which are in a sense simpler objects than the vector fields considered later. Furthermore, in this chapter we use the canonical one dimensional continued fraction algorithm to construct the scaling map, which allows for the determination of an explicit form of the scaling transformations associated to a two-dimensional frequency vector. This is not the case in the higher dimensional situations considered in later chapters.

3.1 Introduction and summary of the results

Here, we define a sequence of renormalization operators $\mathcal{R}_n$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, between Banach spaces of analytic two degree-of-freedom Hamiltonians. The Hamiltonians are functions on complex neighborhoods of $\mathbb{T}^2 \times \{0\}$, where $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ and 0 is the zero vector in $\mathbb{R}^2$. We call these neighborhoods the phase space. We refer to the coordinates of this space, q and p, as angles and momenta, respectively.

Each of the Banach spaces, indexed by n, contains an integrable Hamiltonian H_n^0 with an invariant torus of a given frequency $\omega_n \in \mathbb{R}^2 \backslash \{0\}$ at $p = 0$ and frequencies of nearby tori *twisted* in the direction of $\Omega_n \in \mathbb{R}^2 \backslash \{0\}$. The vectors ω_n and Ω_n are uniquely determined given a pair of vectors $(\omega, \Omega) = (\omega_0, \Omega_0)$. The domain of the n-th step renormalization operator $\mathcal{R}_n$ will be restricted to a small neighborhood of the integrable Hamiltonian H_n^0, keeping the analysis within the scope of classical KAM theory. The n-th step renormalization operator $\mathcal{R}_n$ is a function from that neighborhood into another that maps a Hamiltonian H_n, close to H_n^0, into $H_{n+1} = \mathcal{R}_n(H_n)$, close to $H_{n+1}^0 = \mathcal{R}_n(H_n^0)$. On the winding ratio $\alpha_n = \omega_{n2}/\omega_{n1}$, $\omega_{n1} \neq 0$, of the frequency vector $\omega_n = (\omega_{n1}, \omega_{n2})^\star$, the operator $\mathcal{R}_n$ acts as a shift of its continued fraction expansion. The renormalization of a Hamiltonian H_0 consists of successive application of the operators $\mathcal{R}_n$, $n \in \mathbb{N}_0$. Given a Hamiltonian H_0, we call the sequence of Hamiltonians H_n, $n \in \mathbb{N}_0$, consisting of H_0 and its images $H_{n+1} = \mathcal{R}_n \circ \cdots \circ \mathcal{R}_0(H_0)$, the orbit of the Hamiltonian H_0.

Renormalization techniques in dynamical systems are designed to study systems on progressively smaller spatial scales and longer time scales. The renormalization scheme for Hamiltonians is essentially a transformation of classical Hamiltonian functions generated by a scaling of the phase space $\mathcal{T}_\mu$, modulo a group of transformations G that preserve the topological characteristics of the orbits of the Hamiltonian flows (Figure 3.1). This set includes canonical transformations homotopic to the identity, scaling of the momenta, time and energy. The n-th step renormalization operator basically consists of time-rescaling, composed with a nonlinear diffeomorphism homotopic to the identity, and a linear scaling of the (lifted)

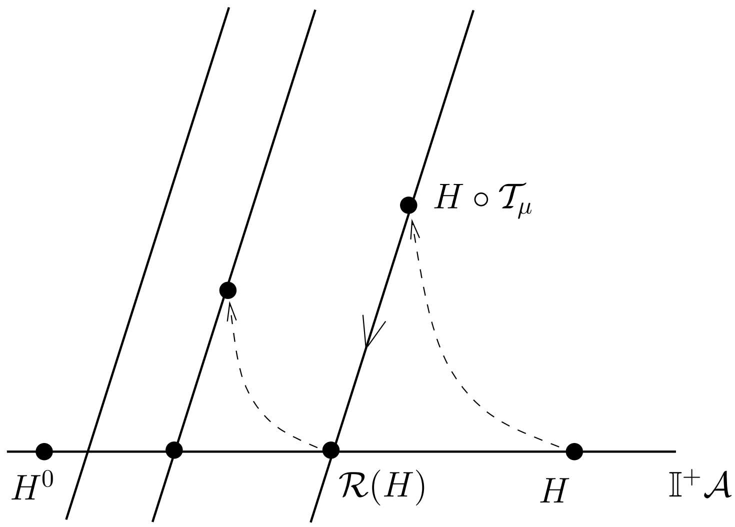

Figure 3.1: *Scaling and elimination in the renormalization of Hamiltonians for the golden-mean frequency ratio. The inclined lines represent the orbits of G.*

phase space.

The linear scaling transformation is intended to enlarge a region around the orbits of the integrable Hamiltonian flow while keeping the periodicity of the angle coordinates. It is a composition of a linear scaling of the momentum space and a linear canonical transformation of the phase space generated by a point transformation in $GL(2, \mathbb{Z})$.

The growth of the Fourier modes of the Hamiltonian in the direction of the dominant flow is prevented by eliminating these modes in each step. This is achieved by a nonlinear canonical transformation homotopic to the identity. The process of elimination (of "irrelevant" modes of a Hamiltonian) and rescaling (of the Fourier lattice) is similar in spirit to block spin transformations in statistical mechanics - a standard tool in the theory of critical phenomena.

Additionally, time-rescaling, a nonlinear scaling of the momenta and a translation in momentum space are included. The latter transformation prevents the

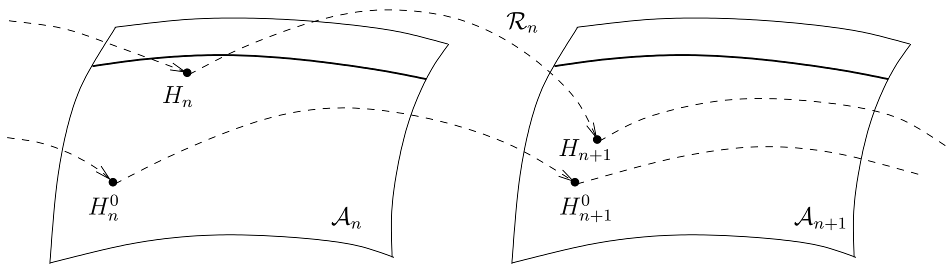

Figure 3.2: *The renormalization of Hamiltonians for Diophantine frequency vectors. An orbit approaches the integrable limit set.*

renormalization operators from having an expanding eigendirection. These transformations are also homotopic to the identity and do not change the winding number of the orbits of the Hamiltonian flow.

The main result of this chapter concerning the construction of the renormalization scheme for analytic Hamiltonians is the following theorem which is a direct consequence of Theorem 3.4.4 and Theorem 3.6.4, that will proved later.

Theorem 3.1.1 *There exists a sequence of renormalization operators* $\{\mathcal{R}_n\}$, $n \in \mathbb{N}_0$, *such that the following holds. The* n*-th step renormalization operator* $\mathcal{R}_n$ *is a well-defined analytic map on an open ball of analytic Hamiltonians around* H_n^0 *for a generic frequency vector* $\omega_n \in \mathbb{R}^2 \backslash \{0\}$*. For a set of Diophantine frequency vectors* ω_0 *of full Lebesgue measure, the renormalization orbits of all Hamiltonians in a neighborhood of an integrable Hamiltonian* H_0^0 *associated to* ω_0*, approach the orbit of the integrable Hamiltonian* H_0^0*.*

As an application of the scheme we prove a classical KAM-type theorem.

Theorem 3.1.2 *Every Hamiltonian* H_0 *sufficiently close to such a* H_0^0 *has an analytic invariant torus on which the motion is conjugate to a linear flow of Diophantine frequency vector* ω_0*.*

The proof of this theorem follows from Theorem 3.7.7 Theorem 3.7.11.

The above-mentioned set of Diophantine frequency vectors for which the convergence result has been obtained and invariant tori have been constructed contains those with Diophantine exponent $\beta < (\sqrt{161} - 11)/10$. Though this set is not optimal, it is of full Lebesgue measure. This is not a restriction of the renormalization method and this set can be extended to all Diophantine frequency vectors. Indeed, this has been done in the later chapters.

This chapter is organized as follows. In the next section, we define precisely the Banach spaces of Hamiltonians that will be renormalized. In Section 3.3, we describe the performed scaling of the phase space. Section 3.4 contains the construction of the n-th step renormalization operator. The existence of a canonical transformation that eliminates the non-resonant modes of a near-integrable Hamiltonian is proved in Section 3.5. In Section 3.6, we prove the convergence of the renormalization dynamics to an integrable limit set for ω satisfying a Diophantine condition. Section 3.7 contains an application of the previously constructed renormalization scheme to the proof of a KAM theorem. We construct the analytic invariant tori with Diophantine frequency vectors for near-integrable Hamiltonians.

3.2 The spaces of Hamiltonians

Let us start by defining the spaces of Hamiltonians that we will consider. Given $n \in \mathbb{N}_0$, let $\omega_n, \Omega_n \in \mathbb{R}^2 \backslash \{0\}$ be two vectors not parallel to each other. We introduce the normalized vector $\hat{\omega}_n = \omega_n / \|\omega_n\|$. Here, and in the remaining part of this chapter, $\|\cdot\|$ denotes ℓ^1-norm of a vector. Define ω'_n and Ω'_n in $\mathbb{R}^2$, by the following relations: $\omega'_n \cdot \Omega_n = 0$, $\omega'_n \cdot \hat{\omega}_n = 1$, $\Omega'_n \cdot \hat{\omega}_n = 0$, and $\Omega'_n \cdot \Omega_n = 1$.

Definition 3.2.1 *Given a pair of positive numbers $\rho = (\rho_1, \rho_2)$, define*

$$\begin{aligned} D_{n,1}(\rho_1) &= \{q \in \mathbb{C}^2 : |\mathrm{Im}\,\omega_n' \cdot q| < \rho_1,\ |\mathrm{Im}\,\Omega_n' \cdot q| < \rho_1\}\,, \\ D_{n,2}(\rho_2) &= \{p \in \mathbb{C}^2 : |\hat{\omega}_n \cdot p| < \rho_2,\ |\Omega_n \cdot p| < \rho_2\}\,, \end{aligned} \tag{3.2.1}$$

and let $D_n(\rho) = D_{n,1}(\rho_1) \times D_{n,2}(\rho_2)$.

The Hamiltonians are analytic functions $H : D_n(\rho) \to \mathbb{C}$, which are 2π-periodic in both q-variables. These functions can be expanded in Fourier-Taylor series

$$H(q,p) = \sum_{(\nu,\kappa)\in I} H_{\nu,\kappa}(\hat{\omega}_n \cdot p)^{\kappa_1}(\Omega_n \cdot p)^{\kappa_2} e^{iq\cdot\nu}\,, \tag{3.2.2}$$

where $\kappa = (\kappa_1, \kappa_2)$ and $I = \mathbb{Z}^2 \times \mathbb{N}_0^2$. We will refer to each term in this sum as a mode of the Hamiltonian H.

Definition 3.2.2 *Given $\rho > 0$, componentwise, define $\mathcal{A}_n(\rho)$ to be the Banach space of functions H that are analytic on $D_n(\rho)$, extend continuously to the boundary of $D_n(\rho)$, and have finite norm*

$$\|H\|_{n,\rho} = \sum_{(\nu,\kappa)\in I} |H_{\nu,\kappa}| \rho_2^{\|\kappa\|} e^{\rho_1(|\hat{\omega}_n\cdot\nu| + |\Omega_n\cdot\nu|)}\,. \tag{3.2.3}$$

Let us also define the projection operators $\mathbb{P}_n^\kappa$ on $\mathcal{A}_n(\rho)$ by

$$\mathbb{P}_n^\kappa H = H_{0,\kappa}(\hat{\omega}_n \cdot p)^{\kappa_1}(\Omega_n \cdot p)^{\kappa_2}, \tag{3.2.4}$$

and let $\mathbb{E} = \sum_{\kappa\in\mathbb{N}_0^2} \mathbb{P}_n^\kappa$ be the projection operator onto the subspace of q-independent Hamiltonians. Define the functionals $\mathfrak{p}_n^\kappa : \mathcal{A}_n(\rho) \to \mathbb{C}$, by $\mathfrak{p}_n^\kappa H = H_{0,\kappa}$.

The analysis of this chapter is focused on Hamiltonians of the form $H = H_n^0 + h$, where

$$H_n^0 = \omega_n \cdot p + \frac{1}{2}(\Omega_n \cdot p)^2, \tag{3.2.5}$$

is an integrable Hamiltonian and $h \in \mathcal{A}_n(\rho)$ is a perturbation. The integrable Hamiltonians H_n^0 are degenerate, in the sense that they do not satisfy Kolmogorov's non-degeneracy condition, but they do have a *twist* in the Ω_n direction, i.e. they satisfy

$$\det\left[\frac{\partial^2 H_n^0}{\partial p_i \partial p_j}\right] = 0, \qquad\qquad \left|\frac{\partial^2 H_n^0}{\partial(\Omega_n \cdot p)^2}\right| = 1 \neq 0. \tag{3.2.6}$$

Hamiltonians satisfying this weaker (than Kolmogorov's original) non-degeneracy condition have been included in the improved versions of the KAM theory.

The sequence of renormalization transformations is associated to the sequence of vector pairs (ω_n, Ω_n), $n \in \mathbb{N}_0$. This sequence has been constructed from a pair of vectors $\omega, \Omega \in \mathbb{R}^2 \backslash \{0\}$. We assume that $\omega \in \mathbb{R}^2$ is of the form $\omega = \ell(1, \alpha)^\star$, where $\ell \in \mathbb{R}^+$ and $\alpha > 1$ is an irrational number. Here "$\star$" stands for "transpose".

The unique continued fraction expansion [34] of such an irrational $\alpha \in \mathbb{R}$ is given by

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots}}\,, \tag{3.2.7}$$

where $a_n \in \mathbb{Z}^+$, $n \geq 0$, are called partial quotients. We also write $\alpha = [a_0, a_1, \ldots]$. The vectors ω_n are constructed from the continued fraction expansion of the winding ratio α of the frequency vector ω. We define $\omega_n = \ell(1, \alpha_n)^\star$, where $\alpha_n = [a_n, a_{n+1}, \ldots]$.

We choose a vector $\Omega \in \mathbb{R}^2$ which is suitable in the sense of the following definition.

Definition 3.2.3 *$\Omega \in \mathbb{R}^2$ will be called suitable if it is of the form $\Omega = (\Omega_1, \Omega_2)^\star$, with $|\Omega_1| \geq |\Omega_2| \geq 0$, $\Omega_1 \Omega_2 \leq 0$ and $\|\Omega\| = 1$.*

Notice that a suitable vector Ω cannot be arbitrary close to the direction of ω. The smaller (positive) angle between these vectors is larger than $\pi/4$ and smaller than $3\pi/4$ radians.

3.3 Scaling of the phase space

In this section, we discuss the performed scaling of the phase space and show that its pullback is analyticity improving.

3.3.1 Continued fractions and the scale change

The renormalization scheme for d degree-of-freedom Hamiltonians introduced by Koch is associated to a frequency vector ω whose components span an algebraic number field of degree $d \in \mathbb{N}$. For two degree-of-freedom systems, that scheme can be applied to Hamiltonians associated to a frequency vector with a quadratic irrational slope. In that case, one can find a hyperbolic matrix $T \in GL(2, \mathbb{Z})$ with determinant ± 1, for which ω is an expanding eigenvector (Lemma 4.1 in [44]). That matrix is then used to perform linear scaling of the phase space at each renormalization step.

In order to construct a renormalization scheme which applies to a larger set of frequency vectors, we use a different linear scaling transformation at each step. We generate the n-th step scaling transformation of the phase space using the matrix $T_n \in GL(2, \mathbb{Z})$ defined by

$$T_n = \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}. \tag{3.3.1}$$

More precisely, at the n-th renormalization step, the scaling of the (lifted) phase space is performed with a linear map $\mathcal{T}_n^{-1}$ on $\mathbb{C}^2 \times \mathbb{C}^2$, defined by $\mathcal{T}_n(q,p) = (T_n q, \mu_n T_n^{\star -1} p)$, where $\mu_n = \alpha_{n+1}^{-1} \|T_n^{-1}\Omega_n\|^{-2}$ is a positive number. This map generates the transformation of Hamiltonians $H_n \mapsto H_n \circ \mathcal{T}_n$. Notice that the vectors ω_n could also be defined recursively using this matrix, via $\omega_{n+1} = \alpha_{n+1} T_n^{-1} \omega_n$, given $\omega = \omega_0$. Similarly, given a suitable vector $\Omega = \Omega_0$, the vectors Ω_n are defined recursively by $\Omega_n = T_{n-1}^{-1}\Omega_{n-1}/\|T_{n-1}^{-1}\Omega_{n-1}\|$, for $n \in \mathbb{N}$. Thus, under this scaling, the integrable Hamiltonian H_n^0 is mapped into $\alpha_{n+1}^{-1}\mu_n H_{n+1}^0$, $n \in \mathbb{N}_0$. A time rescaling, then, normalizes the latter to H_{n+1}^0.

The matrix T_n has the following properties. If $\alpha > 1$, then, for all $n \geq 0$, $a_n \geq 1$ and T_n is a hyperbolic matrix with $\det(T_n) = -1$ and eigenvalues λ_n and $-1/\lambda_n$, where

$$\lambda_n = \frac{a_n + \sqrt{a_n^2 + 4}}{2}\,. \tag{3.3.2}$$

The expanding and contracting eigendirections corresponding to these eigenvalues are given by $(1, \lambda_n)^\star$ and $(1, -1/\lambda_n)^\star$, respectively. The expanding eigenvector is close to ω_n, in the sense that the absolute value of the angle between it and ω_n is smaller than $\pi/4$.

The map $\omega_n \mapsto \omega_{n+1}$ is related with the Gauss map of the fractional part of α_n. Given $\alpha \in \mathbb{R}$, let $[\alpha]$ be the integer part of α, i.e. $[\alpha] = \max\{k \in \mathbb{Z} : k \leq \alpha\}$. Also, let $\{\alpha\} = \alpha - [\alpha]$ be the fractional part of α.

For $x > 0$, the Gauss map is defined as

$$G : x \mapsto \left\{\frac{1}{x}\right\}. \tag{3.3.3}$$

If we define $x_n = \alpha_n - a_n$, for $n \geq 0$, we obtain $x_{n+1} = G(x_n)$. Thus, the shift $[a_n, a_{n+1}, \dots] \mapsto [a_{n+1}, a_{n+2}, \dots]$ of the continued fraction expansion of α_n corresponds to the Gauss map of its fractional part x_n.

The transformation $\alpha_n \mapsto \alpha_{n+1} = (\alpha_n - a_n)^{-1}$ is a special case of a modular transformation,

$$\alpha \mapsto \frac{T_{2,1} + \alpha T_{2,2}}{T_{1,1} + \alpha T_{1,2}}\,, \tag{3.3.4}$$

generated by the action of an integer matrix $T = (T_{i,j}) \in GL(2, \mathbb{Z})$ of determinant ± 1. Numbers related by such a transformation are called equivalent. Two numbers are equivalent if and only if they have the same tail in their continued fraction expansions (Theorem 175 in [34]). The matrix T_n generates an equivalence relation between α_{n+1} and α_n.

The linear coordinate change $\mathcal{T}_n^{-1}$, maps the orbit of H_n^0 with rotation number α_n into the orbit of H_{n+1}^0 with the rotation number α_{n+1}. We have made the particular choice of T_n in order to provide the desired "scale change" in the sense explained below.

Let us consider the sequence of periodic orbits approximating an invariant torus with frequency ratio α. Recursively, define the sequence of convergent matrices associated to α,

$$P_n = \begin{bmatrix} q_{n-1} & p_{n-1} \\ q_n & p_n \end{bmatrix}, \tag{3.3.5}$$

for $n \geq 0$, via $P_n = T_n P_{n-1}$, with P_{-1} being the identity matrix in $GL(2, \mathbb{Z})$. Thus, q_n and p_n are determined by the following recursion relations

$$\begin{aligned} q_n &= a_n q_{n-1} + q_{n-2}, \\ p_n &= a_n p_{n-1} + p_{n-2}, \end{aligned} \tag{3.3.6}$$

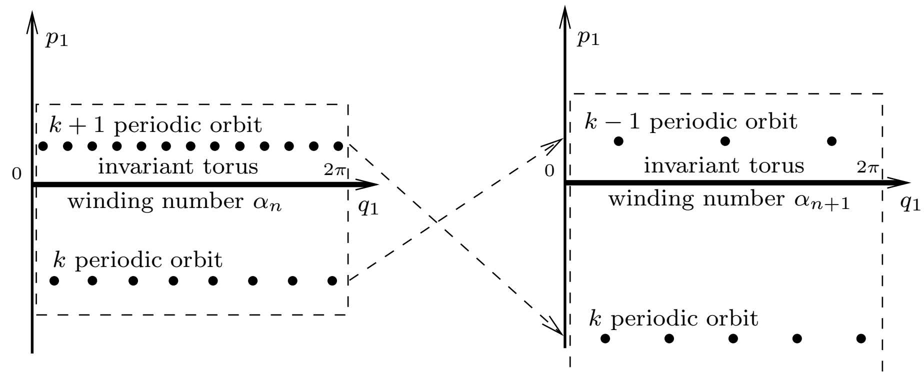

Figure 3.3: *Periodic orbits approximating an invariant torus and a scale change. The numerical values are* $\alpha_n = \gamma + 1$*,* $\alpha_{n+1} = \gamma$*,* $k = 3$*. Here* $\gamma = [1, 1, \dots]$ *is the golden-mean.*

and $q_{-2} = 1, p_{-2} = 0, q_{-1} = 0, p_{-1} = 1$. The convergent matrices define a sequence of convergents $p_n/q_n = [a_0, \dots, a_n]$ that approaches α as $n \to \infty$.

The values of q_n and p_n are associated to α. In the following, we will stress this fact by writing explicitly $q_n(\alpha)$ and $p_n(\alpha)$. Consider the sequence of periodic orbits of frequency ratios $\frac{p_k(\alpha_n)}{q_k(\alpha_n)}$. The matrix T_n produces the following "scale change"

$$\begin{pmatrix} -a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_k(\alpha_n) \\ p_k(\alpha_n) \end{pmatrix} = \begin{pmatrix} q_{k-1}(\alpha_{n+1}) \\ p_{k-1}(\alpha_{n+1}) \end{pmatrix}, \tag{3.3.7}$$

motivating its use. A periodic orbit labelled by k that approximates an invariant torus with frequency ratio α_n of H_n^0 is mapped into a periodic orbit labelled by $k-1$ that approximates an invariant torus with frequency ratio α_{n+1} of H_{n+1}^0.

3.3.2 Resonant modes and analyticity improving property of the scaling transformation

The composition $H_n \circ \mathcal{T}_n$ represents a singular operator, as the matrix T_n has an expanding eigendirection. The composition is, however, harmless when the domain of the operator is restricted to Hamiltonians which contain only the modes for which $\|\kappa\|$ is large or ν is almost perpendicular to ω. These modes are called *resonant*. These are the modes that produce small denominators in KAM theory.

Definition 3.3.1 *Given $\sigma, \varkappa > 0$ and vectors $\omega_n, \Omega_n \in \mathbb{R}^2$, we define the non-resonant index set,*

$$I_n^- = \{(\nu, \kappa) \in I : |\omega_n \cdot \nu| > \sigma|\Omega_n \cdot \nu|,\ |\omega_n \cdot \nu| > \varkappa\|\kappa\|\}. \tag{3.3.8}$$

The resonant index set is defined as its complement, $I_n^+ = I \backslash I_n^-$. The corresponding projection operators on $\mathcal{A}_n(\rho)$, $\mathbb{I}_n^-$ and $\mathbb{I}_n^+ = \mathbb{I} - \mathbb{I}_n^-$, are defined by setting

$$(\mathbb{I}_n^- H)(q, p) = \sum_{(\nu,\kappa)\in I_n^-} H_{\nu,\kappa}(\hat{\omega}_n \cdot p)^{\kappa_1}(\Omega_n \cdot p)^{\kappa_2} e^{iq\cdot\nu}. \tag{3.3.9}$$

Hamiltonians consisting only of resonant modes, will be called resonant. On the subspace of resonant Hamiltonians in $\mathcal{A}_n(\rho)$, the map $H \mapsto H \circ \mathcal{T}_n$ is analyticity improving.

Proposition 3.3.2 *Let $0 < \rho' < \rho$ with $3\rho' > 2\rho$, componentwise. If σ and $\varkappa$ are sufficiently small constants, then every Hamiltonian $H \in \mathbb{I}_n^+ \mathcal{A}_n(\rho')$, $n \in \mathbb{N}_0$, has an analytic extension to $\mathcal{T}_n D_{n+1}(\rho)$. The linear map from $\mathbb{I}_n^+ \mathcal{A}_n(\rho')$ to $\mathcal{A}_{n+1}(\rho)$, given by : $H \mapsto H \circ \mathcal{T}_n$, is compact.*

Proof: Since

$$H \circ \mathcal{T}_n(q,p) = \sum_{(\nu,\kappa)\in I_n^+} H_{\nu,\kappa}(\mu_n T_n^{-1}\hat{\omega}_n \cdot p)^{\kappa_1}(\mu_n T_n^{-1}\Omega_n \cdot p)^{\kappa_2} e^{iq\cdot T^\star \nu}, \qquad (3.3.10)$$

for $H \in \mathbb{I}_n^+ \mathcal{A}_n(\rho')$, one can obtain

$$\begin{aligned}
\|H \circ \mathcal{T}_n\|_{n+1,\rho} &= \sum_{(\nu,\kappa)\in I_n^+} |H_{\nu,\kappa}| \left(\frac{\mu_n\|\omega_{n+1}\|}{\alpha_{n+1}\|\omega_n\|}\rho_2\right)^{\kappa_1} (\mu_n\|T_n^{-1}\Omega_n\|\rho_2)^{\kappa_2} \\
&\quad \cdot\ e^{\rho_1\left(\frac{\alpha_{n+1}}{\|\omega_{n+1}\|}|\omega_n\cdot\nu| + \frac{|\Omega_n\cdot\nu|}{\|T_n^{-1}\Omega_n\|}\right)} \\
&= \sum_{(\nu,\kappa)\in I_n^+} |H_{\nu,\kappa}| \left(\frac{\mu_n\|\omega_{n+1}\|\rho_2}{\alpha_{n+1}\|\omega_n\|\rho_2'}\right)^{\kappa_1} \left(\frac{\mu_n\|T_n^{-1}\Omega_n\|\rho_2}{\rho_2'}\right)^{\kappa_2} \rho_2'^{\|\kappa\|} \\
&\quad \cdot\ e^{\left(\frac{\rho_1\alpha_{n+1}\|\omega_n\|}{\|\omega_{n+1}\|}-\rho_1'\right)|\hat{\omega}_n\cdot\nu| + \left(\frac{\rho_1}{\|T_n^{-1}\Omega_n\|}-\rho_1'\right)|\Omega_n\cdot\nu|} e^{\rho_1'(|\hat{\omega}_n\cdot\nu|+|\Omega_n\cdot\nu|)} \\
&\le \|H\|_{n,\rho'}\,,
\end{aligned}$$

$$(3.3.11)$$

provided that all of the modes contract. The terms with ν obeying the inequality $|\omega_n \cdot \nu| \le \sigma|\Omega_n\cdot\nu|$ contract for sufficiently small σ satisfying the second part of the double inequality

$$\frac{\sigma\frac{\alpha_{n+1}}{\|\omega_{n+1}\|} + \frac{1}{\|T_n^{-1}\Omega_n\|}}{1+\frac{\sigma}{\|\omega_n\|}} \le \frac{\sigma}{\ell} + \frac{2}{3} < \frac{\rho_1'}{\rho_1}\,. \qquad (3.3.12)$$

Given $\sigma > 0$, the first part is satisfied for any $n \in \mathbb{N}_0$. The other conditions are trivially satisfied as $\mu_n\|T_n^{-1}\Omega_n\| \le 2/3$.

The modes indexed by (ν,κ) satisfying $|\omega_n\cdot\nu| \le \varkappa\|\kappa\|$ also contract if $\varkappa$ satisfies the second part of the double inequality

$$\mu_n\|T_n^{-1}\Omega_n\| e^{\varkappa\left(\frac{\rho_1\alpha_{n+1}}{\|\omega_{n+1}\|}-\frac{\rho_1'}{\|\omega_n\|}\right)} \le \frac{2}{3}e^{\varkappa\rho_1/\ell} < \frac{\rho_2'}{\rho_2}\,. \qquad (3.3.13)$$

Given $\varkappa > 0$, the first part is satisfied for any $n \in \mathbb{N}_0$.

These estimates show that $H \circ \mathcal{T}_n$ is analytic in $D_{n+1}(\rho)$. Now, given $\sigma, \varkappa > 0$ satisfying the second parts of the double inequalities (3.3.12) and (3.3.13), one can find $r > \rho$ satisfying $3\rho' > 2r$ componentwise, such that $H \circ \mathcal{T}_n$ is also analytic and bounded in $D_{n+1}(r)$. The assertion follows from the fact that the inclusion map from $\mathcal{A}_{n+1}(r)$ to $\mathcal{A}_{n+1}(\rho)$ is compact. **QED**

3.4 The one-step renormalization operator

We will restrict the domain of the n-th step renormalization operator to resonant Hamiltonians. The composition of a resonant Hamiltonian with $\mathcal{T}_n$ produces, in general, non-resonant modes. In the n-th renormalization step, we completely eliminate these modes such that the renormalized Hamiltonians are also resonant. We also include a translation in the variable $\Omega_{n+1} \cdot p$, to prevent the n-th step renormalization operator from having an expanding direction. The existence of the non-zero quadratic (in the components of p) part of the integrable Hamiltonian H_n^0 is essential for the construction of such a transformation. Finally, we perform an additional, nonlinear, scaling of the action (momentum) variables and time, in order to fix the coefficients of the $(\omega_{n+1} \cdot p)$ and $(\Omega_{n+1} \cdot p)^2$ modes of the renormalized Hamiltonians to 1 and 1/2, respectively.

Definition 3.4.1 *The n-th step renormalization operator $\mathcal{R}_n$ is defined (formally) on an open ball in $\mathbb{I}_n^+ \mathcal{A}_n(\rho')$, with $\rho' > 0$, componentwise, by the following action on a Hamiltonian $H \in \mathbb{I}_n^+ \mathcal{A}_n(\rho')$,*

$$\mathcal{R}_n(H) = \frac{\theta_n'}{\mu_n'}(H \circ \Lambda_n - \mathbb{P}_{n+1}^{(0,0)} H \circ \Lambda_n), \tag{3.4.1}$$

where $\Lambda_n = \mathcal{T}_n \circ V_{H'} \circ U_{H''} \circ S_{\tilde{H}}$. Here $H' = (\theta_n/\mu_n) H \circ \mathcal{T}_n$, with $\theta_n = \alpha_{n+1}$, $H'' = H' \circ V_{H'}$, and $\tilde{H} = H'' \circ U_{H''}$. The transformation $V_{H'} : D_{n+1}(\varrho) \to D_{n+1}(\rho)$, where $3/2\rho' > \rho > \varrho > \varrho' > \rho'$, componentwise, represents the translation $V_{H'}(q,p) = (q, p - v_{H'}\Omega'_{n+1})$, with $v_{H'} \in \mathbb{C}$ determined by the equation $\mathfrak{p}^{(0,1)}_{n+1} H' \circ V_{H'} = 0$. The map $U_{H''} : D_{n+1}(\varrho') \to D_{n+1}(\varrho)$ is a canonical transformation that satisfies $\mathbb{I}^{-}_{n+1}(H'' \circ U_{H''}) = 0$. The transformation $S_{\tilde{H}} : D_{n+1}(\rho') \to D_{n+1}(\varrho')$ represents a scaling $S_{\tilde{H}}(q,p) = (q, z_{\tilde{H}} p)$ of the action variables. The scaling parameters θ'_n and μ'_n take the values $\theta'_n = \theta_n \cdot \tau_{\tilde{H}}$ and $\mu'_n = \mu_n \cdot z_{\tilde{H}}$. The parameters $z_{\tilde{H}}, \tau_{\tilde{H}} \in \mathbb{C}$ are determined such that $\mathfrak{p}^{(0,2)}_{n+1}\mathcal{R}_n(H) = 1/2$ and $\mathfrak{p}^{(1,0)}_{n+1}\mathcal{R}_n(H) = 1$.

In the following, we show that the translations and scalings included in the n-th step renormalization operator are well-defined on a sufficiently small open ball around H^0_{n+1}. Notice first that $H^0_{n+1} = \tilde{H}^0_n = H^{0'}_n = H^{0''}_n = \mathcal{R}_n(H^0_n)$. For $n \in \mathbb{N}_0$, $b > 0$ and $\rho > 0$, componentwise, define $B_{n,\rho}(b)$ to be the open ball of radius b, in $\mathcal{A}_n(\rho)$, centered at H^0_n. Define also $B^{+}_{n,\rho}(b)$ to be the open ball in $\mathbb{I}^{+}_n \mathcal{A}_n(\rho)$ of radius b, centered at H^0_n.

Proposition 3.4.2 *Given $\rho_2 > \varrho_2 > 0$ and $\rho_1 = \varrho_1 > 0$, the following holds for a sufficiently small constant $b > 0$. For every Hamiltonian $H \in B_{n+1,\rho}(b)$, $n \in \mathbb{N}_0$, there exists $v_H \in \mathbb{C}$, such that the translation map $V_H : D_{n+1}(\varrho) \to D_{n+1}(\rho)$, is well-defined by $V_H(q,p) = (q, p - v_H \Omega'_{n+1})$, where $\mathfrak{p}^{(0,1)}_{n+1} H \circ V_H = 0$. The derivative of the map $\mathcal{V}_H : \mathcal{A}_{n+1}(\rho) \to \mathcal{A}_{n+1}(\varrho)$, defined by $\mathcal{V}_H(H) = H \circ V_H$, at H^0_{n+1}, is the linear map $D\mathcal{V}(H^0_{n+1}) = \mathbb{I} - \mathbb{P}^{(0,1)}_{n+1}$.*

Proof: Define the function $F : \mathcal{A}_{n+1}(\rho) \times \mathbb{C} \to \mathbb{C}$, by setting $F(H,v) = \mathfrak{p}^{(0,1)}_{n+1} H \circ V$, where $V(q,p) = (q, p - v\Omega'_{n+1})$. The implicit equation $F(H^0_{n+1}, v) = 0$ has a unique solution $v = v_{H^0_{n+1}} = 0$. Moreover, $D_2 F(H^0_{n+1}, v)|_{v=0} = -1 \neq 0$. We can use this fact to solve the implicit equation, $F(H, v) = 0$, for a given Hamiltonian H in a ball $B_{n+1,\rho}(b)$ of sufficiently small, n-independent radius $b > 0$.

The problem of the existence of a solution $v = v_H$ of the implicit equation $F(H, v) = 0$, for a given Hamiltonian $H \in B_{n+1,\rho}(b)$, is equivalent to the problem of the existence of a fixed point of the function $G_H : v \mapsto v + F(H, v)$. Notice that $|G_H(0)| = |h_{0,(0,1)}| < b/\rho_2$. Here, $H = H^0_{n+1} + h$. We will show that, given $\lambda > 1$, for sufficiently small $b > 0$, G_H is a contraction on a ball of radius $\lambda b/\rho_2$.

Let $|v| \leq \lambda b/\rho_2$, $\lambda > 1$. The norms of $G_H(v)$ and $G'_H(v)$ can be bounded by

$$|G_H(v)| \leq \sum_{\kappa_2=1}^{\infty} |h_{0,(0,\kappa_2)}| \kappa_2 |v|^{\kappa_2 - 1} < \frac{b}{\rho_2(1 - \lambda b/\rho_2^2)^2} = \tilde{b}$$

and

$$|G_{H'}(v)| = |1 + D_2 F(H, v)| \leq \sum_{\kappa_2=2}^{\infty} |h_{0,(0,\kappa_2)}| \kappa_2(\kappa_2 - 1)|v|^{\kappa_2 - 2} < \frac{2b}{\rho_2^2(1 - \lambda b/\rho_2^2)^3} \,,$$

respectively. If $b > 0$ is sufficiently small, then

$$\lambda b/\rho_2 < \rho_2 - \varrho_2 \,, \qquad \frac{1}{(1 - \lambda b/\rho_2^2)^2} < \lambda \qquad \text{and} \qquad \frac{2b}{\rho_2^2(1 - \lambda b/\rho_2^2)^3} < 1 - \frac{1}{\lambda} < 1 \,. \tag{3.4.2}$$

These bounds show that for sufficiently small $b > 0$, G_H is a contraction on a closed ball of radius $\tilde{b}$, and thus, has a unique fixed point in that ball. As the bounds (3.4.2) are independent of n, so is b. The first of the bounds (3.4.2) shows that the translation map is well-defined from $D_{n+1}(\varrho)$ to $D_{n+1}(\rho)$ and that $H \circ V_H$ belongs to $\mathcal{A}_{n+1}(\varrho)$. **QED**

The map $H'' \mapsto H'' \circ U_{H''}$ is well-defined on a sufficiently small ball centered at H^0_{n+1}. Theorem 3.5.6, proved in the next section, guarantees that given $\varrho > \varrho' > 0$, componentwise, for every Hamiltonian $H'' \in \mathcal{A}_{n+1}(\varrho)$, sufficiently close to H^0_{n+1}, there exists a canonical map $U_{H''} : D_{n+1}(\varrho') \to D_{n+1}(\varrho)$, that satisfies the equation $\mathbb{I}^-_{n+1}(H'' \circ U_{H''}) = 0$.

Proposition 3.4.3 *Let $\varrho'_2 > \rho'_2 > 0$ and $\varrho'_1 = \rho'_1 > 0$. For sufficiently small constant $b > 0$ and for every Hamiltonian $H \in B_{n+1,\varrho'}(b)$, $n \in \mathbb{N}_0$, there exist $z_H, \tau_H \in \mathbb{C}$ such that the map $\mathfrak{S}_H : H \mapsto (\tau_H/z_H)(H \circ S_H - \mathbb{P}^{(0,0)}_{n+1} H \circ S_H)$, with $S_H(q,p) = (q, z_H p)$, is well-defined from $\mathcal{A}_{n+1}(\varrho')$ to $\mathcal{A}_{n+1}(\rho')$, and satisfies $\mathfrak{p}^{(0,2)}_{n+1}\mathfrak{S}_H(H) = 1/2$ and $\mathfrak{p}^{(1,0)}_{n+1}\mathfrak{S}_H(H) = 1$. The map S_H maps $D_{n+1}(\rho')$ into $D_{n+1}(\varrho')$. The derivative of the map $\mathfrak{S}_H$ at the point H^0_{n+1} is given by $D\mathfrak{S}(H^0_{n+1}) = \mathbb{I} - \mathbb{P}^{(0,0)}_{n+1} - \mathbb{P}^{(1,0)}_{n+1} - \mathbb{P}^{(0,2)}_{n+1}$.*

Proof: Let $H = H_{n+1} + h \in B_{n+1,\varrho'}(b)$. For sufficiently small $b > 0$, the scaling parameters τ_H and z_H that satisfy the equations $\mathfrak{p}^{(0,2)}_{n+1}\mathfrak{S}_H(H) = 1/2$ and $\mathfrak{p}^{(1,0)}_{n+1}\mathfrak{S}_H(H) = 1$, are given by $\tau_H = 1/(1 + h_{0,(1,0)})$ and $z_H = (1 + h_{0,(1,0)})/(1 + 2h_{0,(0,2)})$. We have the bound

$$|z_H| \le |z_H - 1| + 1 \le \frac{|h_{0,(1,0)} - 2h_{0,(0,2)}|}{1 - 2|h_{0,(0,2)}|} + 1 \le \frac{1 + b/\varrho'_2}{1 - 2b/\varrho'^2_2} < \frac{\varrho'_2}{\rho'_2},$$

where the last inequality is satisfied for sufficiently small n-independent constant $b > 0$. The scaling map is well-defined from $D_{n+1}(\rho')$ to $D_{n+1}(\varrho')$ and the resulting Hamiltonian belongs to $\mathcal{A}_{n+1}(\rho')$. **QED**

We can now show that the n-th step renormalization operator is well-defined on an open ball around H^0_n. Later on we will have to show that the composition of the sequence of operators $\mathcal{R}_n$, with $n \in \mathbb{N}_0$ is also well-defined if the ball around H^0_0 is chosen sufficiently small.

Theorem 3.4.4 *Given $\rho_1' > 0$, for sufficiently small $\sigma, \varkappa > 0$ and $\rho_2' > 0$ satisfying $3\rho_2'/2 < \sigma < \ell/3$, there exists a constant $C' > 0$, such that the n-th step renormalization operator $\mathcal{R}_n$ is a well-defined analytic map from an open ball $B^{+}_{n,\rho'}(\zeta_n)$, of radius $\zeta_n = C'/(\alpha_n\alpha_{n+1})^2$, into $\mathbb{I}^{+}_{n+1}\mathcal{A}_{n+1}(\rho')$. Also, $\|\mathcal{R}_n(H) - H^0_{n+1}\|_{n+1,\rho'} \leq \zeta_n^{-1}\|h\|_{n,\rho'}$, for $H = H^0_n + h \in B^{+}_{n,\rho'}(\zeta_n)$, and*

$$\|\mathcal{R}_n(H) - H^0_{n+1} - D\mathcal{R}_n(H^0_n)h\|_{n+1,\rho'} \leq [\zeta_n(\zeta_n - \|h\|_{n,\rho'})]^{-1}\|h\|^2_{n,\rho'}. \tag{3.4.3}$$

Proof: Let $3\rho'/2 > 3\rho'/(2 + 9\rho_2'/(2\ell)) > \rho > \varrho > \varrho' > \rho'$, componentwise. The bound (3.3.11) implies that there exists a constant $b_1 > 0$, such that $(\theta_n/\mu_n)B^{+}_{n,\rho'}(\zeta_n)\circ \mathcal{T}_n \subset B_{n+1,\rho}(b_1 C')$, where $\zeta_n = C'/(\alpha_n\alpha_{n+1})^2$ and $C' > 0$. Proposition 3.4.2 guarantees that for sufficiently small $C' > 0$,

$$\{H'' : H'' = H' \circ V_{H'},\ H' \in B_{n+1,\rho}(b_1 C')\} \subset B_{n+1,\varrho}(b_1 b_2 C'),$$

with $b_2 > 0$. If $C' > 0$ is chosen sufficiently small, there exists (by Theorem 3.5.6) a canonical transformation $U_{H''}$ for Hamiltonians H'' in a neighborhood of H^0_{n+1} containing $B_{n+1,\varrho}(b_1 b_2 C')$, that satisfies the equation $\mathbb{I}^{-}_{n+1}(H'' \circ U_{H''}) = 0$. Furthermore,

$$\{\tilde{H} : \tilde{H} = H'' \circ U_{H''},\ H'' \in B_{n+1,\varrho}(b_1 b_2 C')\} \subset B^{+}_{n+1,\varrho'}(b_1 b_2 b_3 C'),$$

where $b_3 > 0$ is a constant (dependent on σ, $\varkappa$ and ϱ). Finally, Proposition 3.4.3 guarantees that for sufficiently small $C' > 0$,

$$\{\mathfrak{S}_{\tilde{H}}(\tilde{H}) : \tilde{H} \in B^{+}_{n+1,\varrho'}(b_1 b_2 b_3 C')\} \subset B^{+}_{n+1,\rho'}(b_1 b_2 b_3 b_4 C'),$$

with $b_4 > 0$. This shows that for sufficiently small $C' > 0$, the n-th step renormalization operator is well-defined from $B^+_{n,\rho'}(\zeta_n)$ to $\mathbb{I}^+_{n+1}\mathcal{A}_{n+1}(\rho')$. As the composition of analytic maps, it is an analytic map itself.

Define the map $g : z \mapsto \mathcal{R}_n(H^0_n + zh) - H^0_{n+1}$, where $h \in \mathbb{I}^+_n\mathcal{A}_n(\rho')$ is given such that $H = H^0_n + h \in B^+_{n,\rho'}(\zeta_n)$. This map is analytic from an open ball in $\mathbb{C}$, of radius $\zeta_n/\|h\|_{n,\rho'} > 1$, into $\mathbb{I}^+_{n+1}\mathcal{A}_{n+1}(\rho')$. As $g(0) = 0$, we have, by the maximum principle, the bound

$$\|g(1)\|_{n+1,\rho'} \leq \sup_{\|\bar{h}\|_{n,\rho'}=\zeta_n} \|\mathcal{R}_n(H^0_n + \bar{h}) - H^0_{n+1}\|_{n+1,\rho'} \frac{\|h\|_{n,\rho'}}{\zeta_n}.$$

From the construction of the renormalization operator, $\sup_{\|\bar{h}\|_{n,\rho'}=\zeta_n} \|\mathcal{R}_n(H^0_n + \bar{h}) - H^0_{n+1}\|_{n+1,\rho'}$ can be bounded by a constant $b_1b_2b_3b_4C'$, less than or equal to 1, if $C' > 0$ is sufficiently small. Then, we have $\|\mathcal{R}_n(H^0_n + h) - H^0_{n+1}\|_{n+1,\rho'} \leq \|h\|_{n,\rho'}/\zeta_n$. Cauchy's formula gives an estimate on the norm of the second-order remainder of the Taylor expansion of $\mathcal{R}_n$ about H^0_n,

$$\begin{aligned} F^2_n = \|g(1) - g(0) - g'(0)\|_{n+1,\rho'} &\leq \frac{1}{2\pi} \oint_{|z|=\zeta_n/\|h\|_{n,\rho'}} \frac{\|g(z)\|_{n+1,\rho'}}{|z^2(z-1)|} dz \\ &\leq \frac{\|h\|^2_{n,\rho'}}{\zeta_n(\zeta_n - \|h\|_{n,\rho'})}, \end{aligned} \tag{3.4.4}$$

for sufficiently small $C' > 0$. This provides the second desired bound. **QED**

Remark 3.4.5 *The renormalization operator $\mathcal{R}_n$ is actually analyticity improving and can be defined from an open ball in $\mathbb{I}^+_n\mathcal{A}_n(\rho'')$ into $\mathbb{I}^+_n\mathcal{A}_n(\rho')$, with $\rho' > \rho''$, componentwise. The loss of analyticity in the transformations close to the identity can be reduced by restricting the domain of the renormalization operator to a smaller ball.*

Remark 3.4.6 *The n-th step renormalization operator is well-defined on a space of resonant Hamiltonians. For a given $n \in \mathbb{N}$, this is not a restriction as, by construction, the renormalized Hamiltonians are always resonant. In order to apply the 0-th step renormalization operator to more general Hamiltonians, one can include a pre-renormalization step consisting of a canonical transformation that eliminates their non-resonant modes.*

3.5 Elimination of non-resonant modes

In this section, we construct a canonical transformation that eliminates non-resonant modes of a near-integrable Hamiltonian. The whole construction is associated to a single renormalization step. The index of the renormalization step will be suppressed in this section, in order to simplify the notation. For the construction of the canonical transformation, we follow an approach as in reference [45]. Our Hamiltonians are, however, assumed to be close to an integrable Hamiltonian that contains a term quadratic in momenta. Technically, in the present scheme, one also needs to assure that the elimination of non-resonant modes is possible at each renormalization step. In that context, we emphasize that the constants that appear in this section will be chosen independently of the renormalization step.

We begin by making a canonical change of coordinates $(q,p) \to (x,y)$ with $x_1 = \omega' \cdot q$, $x_2 = \Omega' \cdot q$, $y_1 = \hat{\omega} \cdot p$ and $y_2 = \Omega \cdot p$. We will simplify the notation further, by writing $H(x,y)$ instead of $H(q(x,y),p(x,y))$. Moreover, some of the symbols in this section will have different meaning than in other sections. The use of those symbols should be restricted to this section.

In the new coordinates, the Fourier-Taylor series and the norm of a function

$H \in \mathcal{A}(\rho)$, analytic on

$$D(\rho) = \{x \in \mathbb{C}^2 : |\operatorname{Im} x_1| < \rho_1,\ |\operatorname{Im} x_2| < \rho_1\} \times \{y \in \mathbb{C}^2 : |y_1| < \rho_2,\ |y_2| < \rho_2\}\,, \tag{3.5.1}$$

where $\rho = (\rho_1, \rho_2) > 0$, componentwise, are given by

$$H(x,y) = \sum_{(v,\kappa)\in I} H_{v,\kappa} y_1^{\kappa_1} y_2^{\kappa_2} e^{iv\cdot x}\,, \quad \|H\|_\rho = \sum_{(v,k)\in I} |H_{v,k}|\, \rho_2^{\kappa_1+\kappa_2} e^{\rho_1(|v_1|+|v_2|)}\,. \tag{3.5.2}$$

Here $I = \mathbb{M}^2 \times \mathbb{N}_0^2$, where $\mathbb{M}^2 = \{(\hat{\omega}\cdot\nu, \Omega\cdot\nu) \in \mathbb{R}^2 : \nu \in \mathbb{Z}^2\} \subset \mathbb{R}^2$ is a set bijective to $\mathbb{Z}^2$ that can be determined from the vectors ω and Ω.

The non-resonant index set is defined as

$$I^{-} = \{(v,k) \in I : |v_1| > \frac{\sigma}{\|\omega\|}|v_2|\ \ , |v_1| > \frac{\varkappa}{\|\omega\|}\|\kappa\|\}\,. \tag{3.5.3}$$

The resonant index set is its complement $I^{+} = I \backslash I^{-}$.

We state without proof the following technical proposition. In what follows, the norm of the functions $X = (X_1, X_2) \in \mathcal{A}^2(\rho)$ is defined as $\|X\|_\rho = \max\{\|X_1\|_\rho, \|X_2\|_\rho\}$. We denote by $\partial_i H$, for $i = 1,2$, the partial derivatives of $H(x,y)$ with respect to x_1 and x_2, and for $i = 3,4$, the partial derivatives of the same function with respect to y_1 and y_2, respectively.

Proposition 3.5.1 *Let $\rho = (\rho_1, \rho_2)$ and $\delta = (\delta_1, \delta_2)$ be given pairs of positive numbers and let $0 < \delta < \rho$, componentwise. If $f, g, h \in \mathcal{A}(\rho)$, and $X, Y \in \mathcal{A}^2(\rho)$ satisfy $\|X\|_\rho \le \delta_1$ and $\|Y\|_\rho \le \delta_2$, and $U : (x,y) \mapsto (x+X, y+Y)$ is a given change of variables, then*

(i) $|f(x,y)| \le \|f\|_\rho\,,\ \forall (x,y) \in \mathcal{D}(\rho)\,,$

(ii) $fg \in \mathcal{A}(\rho)$ *and* $\|fg\|_\rho \le \|f\|_\rho \|g\|_\rho\,,$

(iii) $\|\partial_i h\|_{\rho-(\delta_1,0)} \leq \delta_1^{-1}\|h\|_\rho$ *for* $i = 1, 2$ *,*

(iv) $\|\partial_j h\|_{\rho-(0,\delta_2)} \leq \delta_2^{-1}\|h\|_\rho$ *for* $j = 3, 4$ *,*

(v) $\|h \circ U\|_\rho \leq \|h\|_{\rho+\delta}$ *.*

Given a Hamiltonian H, close to H^0, our goal is to construct a canonical transformation U_H that satisfies the equation $\mathbb{I}^- H \circ U_H = 0$. Such a transformation is close to the identity as the Hamiltonian H is close to integrable. We would like to perform first a canonical transformation $U : (x, y) \mapsto (x', y')$, generated by a function ϕ as

$$x' = x + \nabla_{y'}\phi(x, y'), \qquad y' = y - \nabla_x \phi(x, y')\,, \tag{3.5.4}$$

where $\nabla_x = (\partial_1, \partial_2)$ and $\nabla_y = (\partial_3, \partial_4)$, which satisfies the linearized version of the above equation, i.e. $\mathbb{I}^-(H + \{H, \phi\}) = 0$. Here, $\{H, \phi\}$ denotes the Poisson bracket of the functions H and ϕ, defined by $\{H, \phi\} = \nabla_x H \cdot \nabla_y \phi - \nabla_x \phi \cdot \nabla_y H$.

We introduce $\psi = \partial_1 \phi$ and define the operators $\mathcal{D}_i = \partial_i \partial_1^{-1}$, for $i = 1, 2, 3, 4$, on $\mathbb{I}^- \mathcal{A}(\rho)$, $\rho > 0$, componentwise.

Proposition 3.5.2 *$\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$ are bounded linear operators on $\mathbb{I}^-\mathcal{A}(\rho)$, with operator norms satisfying*

$$\|\mathcal{D}_2\| \leq \frac{\|\omega\|}{\sigma}, \qquad \|\mathcal{D}_3\| \leq \frac{\|\omega\|}{\varkappa\rho_2}, \qquad \|\mathcal{D}_4\| \leq \frac{\|\omega\|}{\varkappa\rho_2}\,. \tag{3.5.5}$$

Proof: Consider first the action of the operators $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$ on $H_{v,k}(x, y) = y_1^{\kappa_1} y_2^{\kappa_2} e^{iv\cdot x}$ with (v, k) belonging to I^-. We find that $\|\mathcal{D}_2 H_{v,k}\|_\rho \leq \frac{\|\omega\|}{\sigma}\|H_{v,k}\|_\rho$, $\|(\mathcal{D}_3 + \mathcal{D}_4)H_{v,k}\|_\rho \leq \frac{\|\omega\|}{\varkappa\rho_2}\|H_{v,k}\|_\rho$, $\|\mathcal{D}_3 H_{v,k}\|_\rho \leq \frac{\|\omega\|}{\varkappa\rho_2}\|H_{v,k}\|_\rho$, and $\|\mathcal{D}_4 H_{v,k}\|_\rho \leq \frac{\|\omega\|}{\varkappa\rho_2}\|H_{v,k}\|_\rho$. These bounds extend by linearity to the whole $\mathbb{I}^-\mathcal{A}(\rho)$. **QED**

Let $H = H^0 + h$, where $H^0 = \|\omega\| y_1 + y_2^2/2$ and $h \in \mathcal{A}(\varrho)$, with $\varrho > \eta > 0$, componentwise. On $\mathbb{I}^- \mathcal{A}(\varrho - \eta)$, define the operator $L(h)$ by the following action on an arbitrary $\psi \in \mathbb{I}^- \mathcal{A}(\varrho - \eta)$,

$$L(h)\psi = \frac{1}{\|\omega\|}\left(-y_2 \mathcal{D}_2 \psi + \partial_1 h \mathcal{D}_3 \psi + \partial_2 h \mathcal{D}_4 \psi - \partial_3 h \mathcal{D}_1 \psi - \partial_4 h \mathcal{D}_2 \psi\right). \qquad (3.5.6)$$

Using the bounds obtained in Proposition 3.5.1 and Proposition 3.5.2, we find that

$$\|L(h)\psi\|_{\varrho-\eta} \leq \left[\frac{\varrho_2 - \eta_2}{\sigma} + \left(\frac{2}{\varkappa \eta_1 (\varrho_2 - \eta_2)} + \frac{1}{\sigma \eta_2} + \frac{1}{\|\omega\| \eta_2}\right) \|h\|_\varrho\right] \|\psi\|_{\varrho-\eta}. \quad (3.5.7)$$

As $\|\omega\| \geq 2\ell$, if $\varrho_2 - \eta_2 < \sigma$ and $\|h\|_\varrho$ is sufficiently small, then $\|L(h)\psi\|_{\varrho-\eta} \leq A\|\psi\|_{\varrho-\eta}$ for every $\psi \in \mathbb{I}^- \mathcal{A}(\varrho - \eta)$ and some positive constant $A < 1$. Therefore, the operator norm $\|L(h)\| \leq A < 1$.

This enables us to solve the equation $\mathbb{I}^-(H + \{H, \phi\}) = 0$ for the generating function of the canonical transformation U.

Proposition 3.5.3 *Let H belong to $\mathcal{A}(\varrho)$, where $\varrho > \eta > 0$, componentwise, and $\varrho_2 - \eta_2 < \sigma$. If $\|h\|_\varrho$ is sufficiently small such that, by the inequality (3.5.7), $L(h)$ is an operator on $\mathbb{I}^- \mathcal{A}(\varrho - \eta)$ bounded in norm by a positive constant $A < 1$, then the equation*

$$\mathbb{I}^-(H + \{H, \phi\}) = 0, \qquad \mathbb{I}^+ \phi = 0, \qquad (3.5.8)$$

has a unique solution ϕ, such that $\psi = \partial_1 \phi$ belongs to $\mathbb{I}^- \mathcal{A}(\varrho - \eta)$, and satisfies

$$\|\psi\|_{\varrho-\eta} \leq (1 - A)^{-1} \frac{\|\mathbb{I}^- H\|_{\varrho-\eta}}{\|\omega\|}, \quad \|\{H, \phi\}\|_{\varrho-\eta} \leq (1 - A)^{-1} \|\mathbb{I}^- H\|_{\varrho-\eta}. \quad (3.5.9)$$

Proof: Let $L^\pm = \mathbb{I}^\pm L(h) \mathbb{I}^-$. Equation (3.5.8) can now be written in the form $(\mathbb{I} - L^-)\psi = \mathbb{I}^- h/\|\omega\|$. The assumptions guarantee that the operator norms of

$L^{\pm}$ satisfy $\|L^{\pm}\| \leq A < 1$. Equation (3.5.8) can then be solved by inverting the operator $\mathbb{I} - L^{-}$ by means of a Neumann series. The solution ψ satisfies the first of the bounds (3.5.9). As $\{H, \phi\} = (L(h) - \mathbb{I})\psi\|\omega\| = L^{+}\psi\|\omega\| - \mathbb{I}^{-}h$, using this bound one obtains the second one. **QED**

The next proposition shows that the function ϕ generates an explicit canonical transformation.

Proposition 3.5.4 *Let* $0 < \delta_2'' = 2\sigma^{-1}\epsilon^3 < \epsilon r_2 = \delta_2' < \varrho_2 - \eta_2$ *and* $\sigma < 2\ell$. *Let also* $\delta' = (0, \delta_2')$. *Define* B *to be the closed ball of radius* $\delta_2''/2$ *in* $\mathcal{A}(\varrho - \eta - \delta')$ *centered at zero. If* $\psi \in \mathbb{I}^{-}\mathcal{A}(\varrho - \eta)$ *satisfies* $\|\psi\|_{\varrho-\eta} \leq \epsilon^3/\|\omega\|$, *the equation*

$$K(g) = g\,, \qquad K(g) = \nabla_x\phi(x, y - g)\,, \qquad g = g(x,y) = y - y'\,, \tag{3.5.10}$$

has a unique solution $g \in B^2 = B \times B$ *and* $\|g\|_{\varrho-\eta-\delta'} \leq \sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta}$.

Proof: By assumption, for every $g \in B^2$, $\|g\|_{\varrho-\eta-\delta'} \leq \delta_2''/2$. Using the bounds obtained in Proposition 3.5.1 and Proposition 3.5.2, we find that for every $g, g' \in B^2$, there exists $g^* \in B^2$, such that

$$\|K(g)\|_{\varrho-\eta-\delta'} \leq \|\nabla_x\phi\|_{\varrho-\eta} \leq \max\{1, \sigma^{-1}\|\omega\|\}\|\psi\|_{\varrho-\eta} \leq \sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} \leq \sigma^{-1}\epsilon^3 \tag{3.5.11}$$

and

$$\begin{aligned}
&\|K(g') - K(g)\|_{\varrho-\eta-\delta'} \leq \|\nabla_g K(g^*)\|_{\varrho-\eta-\delta'}\|g' - g\|_{\varrho-\eta-\delta'} \\
&\leq \|\nabla_{y'}\nabla_x\phi(x,y')\|_{\varrho-\eta-\delta'}\|g' - g\|_{\varrho-\eta-\delta'} \leq \|\nabla_y\nabla_x\phi\|_{\varrho-\eta-\delta'/2}\|g' - g\|_{\varrho-\eta-\delta'} \\
&\leq 2\delta_2'^{-1}\|\nabla_x\phi\|_{\varrho-\eta}\|g' - g\|_{\varrho-\eta-\delta'} \leq 2\delta_2'^{-1}\max\{1, \sigma^{-1}\|\omega\|\}\|\psi\|_{\varrho-\eta}\|g' - g\|_{\varrho-\eta-\delta'} \\
&\leq 2\delta_2'^{-1}\sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta}\|g' - g\|_{\varrho-\eta-\delta'} \leq 2\delta_2'^{-1}\sigma^{-1}\epsilon^3\|g' - g\|_{\varrho-\eta-\delta'}\,.
\end{aligned}$$

As, by assumption, $2\delta_2'^{-1}\sigma^{-1}\epsilon^3 < 1$, these inequalities show that K is a contraction on B^2, and thus has a unique fixed point $g \in B^2$. The inequality (3.5.11) provides the desired bound on the norm of g. **QED**

Proposition 3.5.5 *Let $\varrho - \eta - 2\delta' > 0$, $\eta > 0$ and $\delta' = (\delta_1', \delta_2') = \epsilon r > 0$, componentwise, with $\delta_1' = \sigma\delta_2'/[\varkappa(\varrho_2 - \eta_2)]$. Also let $0 < \delta_2'' = 2\sigma^{-1}\epsilon^3 < \delta_2' = \epsilon r_2 < \varrho_2 - \eta_2 < \sigma < 2\ell$. Assume further that $H \in \mathcal{A}(\varrho - \eta)$ satisfies $\|H - H^0\|_{\varrho-\eta} < b$ and that $b > 0$ is small enough such that the linear operator $L(H - H^0)$ is bounded on $\mathbb{I}^- \mathcal{A}(\varrho - \eta)$ by $\|L(H - H^0)\| \le A < A' < 1$. If $\|\mathbb{I}^- H\|_{\varrho-\eta} \le (1 - A')\epsilon^3$, the canonical transformation U exists and maps $D(\varrho - \eta - 2\delta')$ into $D(\varrho - \eta)$. The function $H \circ U$ belongs to $\mathcal{A}(\varrho - \eta - 2\delta')$ and $L(H \circ U - H^0)$ is a bounded operator on $\mathbb{I}^- \mathcal{A}(\varrho - \eta - 2\delta')$. They satisfy the bounds*

$$
\begin{aligned}
&\|\mathbb{I}^- (H \circ U)\|_{\varrho-\eta-2\delta'} \le C_2(\epsilon)\epsilon^4\,, \\
&\|H \circ U - H\|_{\varrho-\eta-2\delta'} \le (1 + \epsilon C_2(\epsilon))\epsilon^3 = \Delta b(\epsilon)\,, \qquad (3.5.12)\\
&\|L(H \circ U - H^0)\| \le \|L(H - H^0)\| + C_3(\epsilon)(1 + \epsilon C_1(\epsilon))\epsilon^2 = A + \Delta A(\epsilon)\,,
\end{aligned}
$$

where

$$
C_n(\epsilon) = \frac{\sigma n\epsilon r_2 + \varrho_2 n\epsilon r_2 + \frac{1}{2}(n\epsilon r_2)^2 + \|h\|_{\varrho-\eta}}{\sigma n r_2(\sigma n r_2 - \epsilon^2)}\,,
$$

for $n = 1, 2$, and

$$
C_3(\epsilon) = \frac{2(\varrho_2 - \eta_2)}{\sigma r_2(\varrho_2 - \eta_2 - 2r_2\epsilon)} + \frac{1}{\sigma r_2} + \frac{1}{\ell r_2}\,.
$$

Proof: Our assumptions guarantee that there exists a canonical transformation U with a generating function ϕ that solves the linear equation (3.5.8). More specifically, if $b > 0$ has been chosen sufficiently small, then there exists $\psi \in \mathcal{A}(\varrho - \eta)$ satisfying $\|\psi\|_{\varrho-\eta} < \epsilon^3/\|\omega\|$ and $g \in B^2$ of norm $\|g\|_{\varrho-\eta-\delta'} < \sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta}$ that solves the equation (3.5.10).

Define the following one parameter ($s \in \mathbb{C}$) family

$$
\begin{aligned}
F(s) = -s(\|\omega\|\psi(x, y-sg) + y_2\mathcal{D}_2\psi(x, y-sg)) + \frac{1}{2}s^2(\mathcal{D}_2\psi(x, y-sg))^2 \\
+ h(x + s\nabla_{y-sg}\phi(x, y-sg), y - s\nabla_x\phi(x, y-sg)) \,,
\end{aligned}
$$

passing through $F(0) = H - H^0$ and $F(1) = H \circ U - H^0$, with $F'(0) = \{H, \phi\}$. Assuming that $|s| \le s_0 = \sigma\epsilon^{-3}n\delta_2'$, where $n \in \{1, 2\}$, and using the Proposition 3.5.1 and Proposition 3.5.2, we obtain the following bounds,

$$
\begin{aligned}
&\|sg\|_{\varrho-\eta-n\delta'} \le s_0\sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} < n\delta_2' \,, \\
&\|s\nabla_x\phi(x, y-sg)\|_{\varrho-\eta-n\delta'} \le s_0\sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} < n\delta_2' \,, \\
&\|s\nabla_{y-sg}\phi(x, y-sg)\|_{\varrho-\eta-n\delta'} \le s_0[\varkappa(\varrho_2 - \eta_2)]^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} < n\delta_1' \,.
\end{aligned}
\tag{3.5.13}
$$

These bounds, together with Proposition 3.5.1, imply that $F(s)$ belongs to $\mathcal{A}(\varrho - \eta - n\delta')$, whenever $|s| \le s_0$. In fact, this is true on an open neighborhood of the disc $|s| \le s_0$, as the inequalities (3.5.13) are strict due to $A < A'$. Now, we have

$$
\begin{aligned}
\|H \circ U - H - \{H, \phi\}\|_{\varrho-\eta-n\delta'} &= \|F(1) - F(0) - F'(0)\|_{\varrho-\eta-n\delta'} \\
&= \left\| \frac{1}{2\pi i} \oint_{|s|=s_0} \frac{ds}{s^2(s-1)} F(s) \right\|_{\varrho-\eta-n\delta'} \\
&\le \frac{1}{s_0(s_0-1)}(s_0\|\omega\|\|\psi\|_{\varrho-\eta} + s_0\varrho_2\sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} \\
&\qquad + \frac{1}{2}s_0^2\sigma^{-2}\|\omega\|^2\|\psi\|_{\varrho-\eta}^2 + \|h\|_{\varrho-\eta}) \\
&\le \frac{\sigma n\delta_2' + \varrho_2 n\delta_2' + \frac{1}{2}(n\delta_2')^2 + \|h\|_{\varrho-\eta}}{\sigma\epsilon^{-3}n\delta_2'(\sigma\epsilon^{-3}n\delta_2' - 1)} = C_n(\epsilon)\epsilon^4 \,.
\end{aligned}
$$

As $\mathbb{I}^-(H \circ U) = \mathbb{I}^-(H \circ U - H - \{H, \phi\})$, the first of the inequalities (3.5.12) immediately follows from this estimate for $n = 2$. From Proposition 3.5.3 and the inequality

$$\|H \circ U - H\|_{\varrho-\eta-2\delta'} \leq \|H \circ U - H - \{H, \phi\}\|_{\varrho-\eta-2\delta'} + \|\{H, \phi\}\|_{\varrho-\eta-2\delta'}\,,$$

we find that $\|H \circ U - H\|_{\varrho-\eta-2\delta'} \leq (1-A)^{-1}\|\mathbb{I}^{-} H\|_{\varrho-\eta} + C_2(\epsilon)\epsilon^4$. This implies the second inequality in (3.5.12). The fact that the map U takes $D(\varrho - \eta - 2\delta')$ into $D(\varrho - \eta)$ follows from the bounds (3.5.13) and Proposition 3.5.1.

From the definition of the operator $L(h)$, one can find that

$$\|L(H\circ U-H^0)\| \leq \|L(H-H^0)\|+\left(\frac{2}{\varkappa\delta_1'(\varrho_2-\eta_2-2\delta_2')}+\frac{1}{\sigma\delta_2'}+\frac{1}{\|\omega\|\delta_2'}\right)\|H\circ U-H\|_{\varrho-\eta-\delta'}.$$

Using the inequality $\|H\circ U-H\|_{\varrho-\eta-\delta'} \leq (1+\epsilon C_1(\epsilon))\epsilon^3$, which can be obtained analogously to the second bound in (3.5.12), one can obtain the last desired inequality.
QED

Theorem 3.5.6 *Let $\varrho > \varrho' > 0$, componentwise, $\varrho_2 < \sigma < 2\ell$ and $0 < A' < 1$. Let B be an open set of Hamiltonians $H \in \mathcal{A}(\varrho)$, for which $\|H - H^0\|_{\varrho} < b$ and $\|\mathbb{I}^{-} H\|_{\varrho} < (1 - A')\epsilon^3$. If $b > 0$ and $\epsilon > 0$ are sufficiently small, then for every Hamiltonian $H \in B$ there exists an analytic canonical transformation $U_H : D(\varrho') \to D(\varrho)$ that solves the equation $\mathbb{I}^{-} H \circ U_H = 0$. The map $H \mapsto H \circ U_H$ is analytic from B to $\mathbb{I}^{+}\mathcal{A}(\varrho')$, and*

$$\|H \circ U_H - H\|_{\varrho'} \leq \epsilon^3 + \mathcal{O}(\epsilon^4). \tag{3.5.14}$$

Proof: Let $\varrho > \varrho-\eta > \varrho' > 0$ and $r > 0$, componentwise, with $r_1 = \sigma r_2/[\varkappa(\varrho_2-\eta_2)]$. For sufficiently small $b > 0$, the norm of the operator $L(H - H^0)$ from$\mathbb{I}^{-}\mathcal{A}(\varrho - \eta)$

into $\mathcal{A}(\varrho - \eta)$ is bounded by a positive constant $A < A' < 1$.

By Proposition 3.5.5, there exists a canonical transformation U that maps $D(\varrho-\eta-2\epsilon r)$ into $D(\varrho-\eta)$ such that $\|\mathbb{I}^{-}(H\circ U)\|_{\varrho-\eta-2\epsilon r} \le C_2(\epsilon)\epsilon^4$. We would like to iterate the map $H \mapsto H\circ U$, indefinitely. Introducing $f(\epsilon) = [\epsilon C_2(\epsilon)/(1-A')]^{1/3}\epsilon$, we obtain $\|\mathbb{I}^{-}(H\circ U)\|_{\varrho-\eta-2\epsilon r} \le (1-A')f(\epsilon)^3$. Let $\epsilon_i = f^i(\epsilon)$, for $i \in \mathbb{N}_0$. For sufficiently small ϵ, the sum $\sum_{i=0}^{\infty} 2\epsilon_i r$ converges to a limit $\delta = \mathcal{O}(\epsilon)$. The sums $\sum_{i=0}^{\infty} \Delta b(\epsilon_i)$ and $\sum_{i=0}^{\infty} \Delta A(\epsilon_i)$ converge to $\Delta b = \mathcal{O}(\epsilon^3)$ and $\Delta A = \mathcal{O}(\epsilon^2)$, respectively. Thus, for sufficiently small ϵ the map $(H, \epsilon, \varrho - \eta) \mapsto (H \circ U, f(\epsilon), \varrho - \eta - 2\epsilon r)$ can be iterated indefinitely and the iterations converge to a limit $(H \circ U_H, 0, \varrho - \eta - \delta)$. For sufficiently small ϵ, $\varrho - \eta - \delta > \varrho'$, componentwise.

As ϵ_i are summable, the sequence of canonical transformations U generates a uniformly convergent sequence on $D(\varrho - \eta - \delta)$. The analyticity of the map $H \mapsto H \circ U_H$ follows from uniform convergence of our iteration scheme. The desired bound can be obtained from the second inequality in (3.5.12) and its iterations. **QED**

Let $\hat{H}$ be the Hamiltonian vector field operator generated by H, i.e. $\hat{H} = (\mathbb{J}\nabla H) \cdot \nabla$, where $\mathbb{J}(q,p) = (p, -q)$ and $\nabla = (\nabla_q, \nabla_p)$. In other words, for any phase-space function f, one has $\hat{H} f = \{H, f\}$.

The derivative of the map $\mathcal{N}_H : H \mapsto H \circ U_H$, for $H \in \mathcal{A}_n(\varrho)$, at a resonant Hamiltonian H^+ is given by $D\mathcal{N}(H^+) = \mathbb{I}_n^+ - \mathbb{I}_n^+ \hat{H}^+ (\mathbb{I}_n^- \hat{H}^+ \mathbb{I}_n^-)^{-1} \mathbb{I}_n^-$.

Let $H_n^K = \omega_n \cdot p$, and let $\mathbb{E}H = H_n^K + f$, where $f = \frac{1}{2}(\Omega_n \cdot p)^2 + \mathbb{E}h$. The derivative of the map $\mathcal{N}_H$ at a q-independent Hamiltonian $\mathbb{E}H$ is

$$D\mathcal{N}(\mathbb{E}H) = \mathbb{I}_n^+ - \mathbb{I}_n^+ \hat{f} \hat{H_n^K}^{-1} \mathbb{I}_n^- \left(\mathbb{I} + \mathbb{I}_n^- \hat{f} \hat{H_n^K}^{-1} \mathbb{I}_n^-\right)^{-1} \mathbb{I}_n^-. \tag{3.5.15}$$

The norm of the operator $\hat{f}\hat{H}_n^{K^{-1}}\mathbb{I}_n^- : \mathcal{A}_n(\varrho) \to \mathcal{A}_n(\varrho')$ satisfies the bound

$$\|\hat{f}\hat{H}_n^{K^{-1}}\mathbb{I}_n^-\| \leq \frac{\varrho_2'}{\sigma} + \frac{1}{\|\omega_n\|}\sum_{k\in\mathbb{N}_0^2} |h_{0,k}| \left(\kappa_1 + \kappa_2 \frac{\|\omega_n\|}{\sigma}\right) \varrho_2'^{\|\kappa\|-1}, \tag{3.5.16}$$

and thus

$$\|\hat{f}\hat{H}_n^{K^{-1}}\mathbb{I}_n^-\| \leq \frac{\varrho_2}{\sigma} + \left(\frac{1}{\|\omega_n\|} + \frac{1}{\sigma}\right) \frac{\|\mathbb{E}h\|_{n,\varrho}}{\varrho_2(1-\varrho_2'/\varrho_2)^2}. \tag{3.5.17}$$

As $\varrho_2 < \sigma$, if $\|\mathbb{E}h\|_{n,\varrho}$ is sufficiently small, then $\|\hat{f}\hat{H}_n^{K^{-1}}\mathbb{I}_n^-\| < 1$ and thus the operator $(\mathbb{I}+\mathbb{I}_n^-\hat{f}\hat{H}_n^{K^{-1}}\mathbb{I}_n^-)$ in (3.5.15) can be inverted by means of a Neumann series. If $\varrho_2 < \sigma/2$ and $\|\mathbb{E}h\|_{n,\varrho}$ is sufficiently small, then the operator norm $\|D\mathcal{N}(\mathbb{E}H)\|$ can be bounded by 1.

3.6 Convergence of the renormalization scheme

In this section, we show that the orbits of all Hamiltonians in a neighborhood of an integrable Hamiltonian associated to a Diophantine frequency vector converge to its own orbit under renormalization. We begin by providing some bounds that will be used later on.

3.6.1 Continued fractions and Diophantine bounds

Recall that the n-th step renormalization operator is associated to the pair of vectors (ω_n, Ω_n) generated from a pair (ω, Ω). We assume that $\omega \in \mathbb{R}^2$ is of the form $\omega = \ell(1,\alpha)^\star$, where $\alpha > 1$ and $\ell \in \mathbb{R}^+$. For the vectors $\omega = \ell(1,\alpha)^\star$ with $\alpha < 1$, a canonical transformation of the phase space can be performed, generated by a matrix from $GL(2,\mathbb{Z})$, such that the "new" vector ω is of the desired form. Thus, without loss of generality we can assume that $\alpha > 1$ and, consequently, for any

$n \in \mathbb{N}_0$, $\omega_n = \ell(1, \alpha_n)^\star$ with $\alpha_n > 1$.

The vectors $(1, \alpha_n)^\star = (-1)^n (p_{n-1} - \alpha q_{n-1})^{-1} P_{n-1}^{\star -1} (1, \alpha)^\star$ can be obtained using the convergent matrices (3.3.5). Therefore, we have

$$\alpha = \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}} . \tag{3.6.1}$$

The convergents p_n/q_n satisfy (Theorem 171 in [34])

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} . \tag{3.6.2}$$

They are the best rational approximates of α, in the sense that for any $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, such that $0 < q \leq q_n$ and $p/q \neq p_n/q_n$, $n \in \mathbb{N}$, one has $|p_n - \alpha q_n| < |p - \alpha q|$ (Theorem 182 in [34]).

From the equality (3.6.1), for $n \geq 0$, we find

$$x_n = -\frac{p_n - \alpha q_n}{p_{n-1} - \alpha q_{n-1}} . \tag{3.6.3}$$

Denoting the product of the first $(n+1)$ numbers x_i by $\beta_n = \prod_{i=0}^n x_i = \prod_{i=0}^n \alpha_{i+1}^{-1}$, we obtain that $\beta_n = (-1)^{n+1}(p_n - \alpha q_n)$. Notice that $\det(P_n) = (-1)^{n+1}$. Using the equality (3.6.1) again, one easily finds that $\beta_n^{-1} = q_{n+1} + q_n x_{n+1}$, and thus, $q_{n+1} < \beta_n^{-1} < 2q_{n+1}$.

Define $\tilde{A}_n = \prod_{i=0}^n \alpha_i$. As $\beta_n = \alpha_0 \tilde{A}_{n+1}^{-1}$, the previous bound implies

$$\alpha_0 q_n < \tilde{A}_n < 2\alpha_0 q_n , \tag{3.6.4}$$

assuming that $\alpha_0 > 0$.

Emphasizing again that the values of q_n and p_n are associated to α by writing explicitly $q_n(\alpha)$ and $p_n(\alpha)$, notice that $q_{n+1}(\alpha_0) = p_n(\alpha_1)$. The double inequality (3.6.4) for $(n+1)$ instead of n, can be written as

$$p_n(\alpha_1) < \frac{\tilde{A}_{n+1}}{\alpha_0} < 2p_n(\alpha_1)\,, \tag{3.6.5}$$

implying the following bounds on p_n,

$$p_n < \tilde{A}_n < 2p_n\,. \tag{3.6.6}$$

It is easy to show that $\tilde{A}_n$ grows at least exponentially with n. If $1 < \alpha_i \leq \gamma$, for some $i \in \{0, \ldots, n-1\}$, then $\alpha_i \alpha_{i+1} = \alpha_i/(\alpha_i - a_i) \geq \gamma/(\gamma - 1) = \gamma^2$, where $\gamma = (1+\sqrt{5})/2$ is the golden mean, the limit of the sequence of ratios F_{k+1}/F_k of successive Fibonacci numbers F_k, defined by $F_{k+2} = F_{k+1} + F_k$, for $k \in \mathbb{N}$, and $F_1 = F_2 = 1$. This implies that $\tilde{A}_n/\tilde{A}_{j-1} \geq \gamma^{n-j}$, for $0 < j \leq n$. If $\alpha_0 > 1$, then $\tilde{A}_n \geq \gamma^n$ and the previous inequality is also valid for $j = 0$, with $\tilde{A}_{-1} = 1$.

This growth can be controlled if α is a Diophantine number. An irrational number α will be called Diophantine of order $\beta \geq 0$ if there exists a constant $C > 0$, such that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$,

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^{2+\beta}}\,. \tag{3.6.7}$$

The set of all Diophantine numbers of order β will be denoted by $\mathcal{D}(\beta)$. Sometimes, less precisely, we will call Diophantine a vector $\omega \in \mathbb{R}^2$ with a Diophantine winding ratio.

Definition 3.6.1 *$\omega \in \mathbb{R}^2$ is Diophantine of order β if there exists $C > 0$, such that for all $\nu \in \mathbb{Z}^2 \backslash \{0\}$,*

$$|\omega \cdot \nu| > C\|\nu\|^{-(1+\beta)}. \tag{3.6.8}$$

We will also use the symbol $\mathcal{D}(\beta)$ to denote the set of all Diophantine vectors of order β in $\mathbb{R}^2$.

The Diophantine condition on α, together with the inequality (3.6.2), imposes an upper bound on the growth rate of the denominators of its convergents. For $\alpha \in \mathcal{D}(\beta)$, there exists a constant $K > 0$ such that $q_{n+1} < K q_n^{1+\beta}$, for all $n \in \mathbb{N}_0$. Equivalently, using the inequalities (3.6.4), the Diophantine condition can be written as

$$\tilde{A}_{n+1} < \tilde{K} \tilde{A}_n^{1+\beta}\,, \tag{3.6.9}$$

where $\tilde{K} > 0$ is a constant.

In particular, constant-type numbers, which have a bounded sequence of partial quotients, are Diophantine of order zero. Among them are quadratic irrationals, i.e. the roots of quadratic equations with integer coefficients, whose continued fraction expansions are eventually periodic. Constant-type numbers have zero Lebesgue measure in the real numbers. Diophantine numbers of any order $\beta > 0$ are of measure one.

Though the construction of a one-step renormalization transformation is more general, the bound (3.6.9) plays an essential role in the proof of Theorem 3.6.4 concerning the existence of a trivial attracting orbit of the sequence of renormalization operators which is associated to a Diophantine vector $\omega \in \mathbb{R}^2$. To prove the theorem, we will also need the bounds obtained in the following proposition.

Proposition 3.6.2 *If $\Omega_0 \in \mathbb{R}^2$ is suitable, so is every $\Omega_n = T_{n-1}^{-1}\Omega_{n-1}/\|T_{n-1}^{-1}\Omega_{n-1}\|$, for all $n \in \mathbb{N}$. For a suitable choice of Ω_0, we have the following bounds,*

$$\frac{1+\alpha_0}{4\alpha_0}(\tilde{A}_n + \tilde{A}_{n-1}) < \prod_{i=0}^{n} \|T_i^{-1}\Omega_i\| < (\tilde{A}_n + \tilde{A}_{n-1})\,. \tag{3.6.10}$$

Proof: The first part of the claim can be proved by induction. Notice that $\|P_n^{\star^{-1}}\Omega_0\| = \|T_n^{-1}\dots T_0^{-1}\Omega_0\| = \prod_{i=0}^{n}\|T_i^{-1}\Omega_i\|$, as the matrices T_i are symmetric. One easily finds that for a suitable $\Omega_0 \in \mathbb{R}^2$, $1/2(p_n + p_{n-1} + q_n + q_{n-1}) \leq \|P_n^{\star^{-1}}\Omega_0\| \leq p_n + p_{n-1}$. The bounds (3.6.10) follow from the inequalities (3.6.4) and (3.6.6). **QED**

3.6.2 An attracting limit set

Recall that the orbit of an integrable Hamiltonian $H_0^0 = \omega \cdot p + 1/2(\Omega \cdot p)^2$ under the renormalization consists of Hamiltonians $H_n^0 = \omega_n \cdot p + 1/2(\Omega_n \cdot p)^2$, where $n \in \mathbb{N}_0$. The maps $\omega_n \mapsto \omega_{n+1}$ and $\Omega_n \mapsto \Omega_{n+1}$ are induced by the Gauss map of the inverse winding ratio of ω.

The vectors ω with the winding ratio $\alpha = [a, a, \dots]$, where $a \in \mathbb{N}$, are the fixed points of the first of these maps. A suitable vector Ω is not necessarily a fixed point of the dynamics. However, in this case, it is possible to make a particular choice of Ω, such that it is a fixed point of the map. The corresponding integrable Hamiltonian H_0^0 is then a fixed point of the renormalization.

Similarly, if the winding ratio of a frequency vector ω has a periodic continued fraction expansion, one can make a particular choice of a suitable vector Ω, such that the Hamiltonian H_0^0 generates a periodic orbit of the renormalization. More generally, if the winding ratio of ω is a quadratic irrational and a particular choice of a suitable vector Ω is made, the dynamics of H_0^0 eventually settles on a periodic orbit.

In the following, we show that if ω_0 is a Diophantine vector and Ω_0 is an arbitrary suitable vector, then the orbit of a resonant Hamiltonian H_0, sufficiently close to H_0^0, approaches the orbit of H_0^0 exponentially fast, under the action of the sequence of the renormalization operators $\mathcal{R}_n$, $n \in \mathbb{N}_0$.

We first present a description of the eigenspaces of the derivative of the n-th step renormalization operator at H_n^0, which is given by its action on an arbitrary function $f \in \mathbb{I}_n^+ \mathcal{A}_n(\rho')$, where $\rho' > 0$, componentwise, as

$$D\mathcal{R}_n(H_n^0)f = \frac{\theta_n}{\mu_n}(\mathbb{I} - \mathbb{P}_{n+1}^{(0,0)} - \mathbb{P}_{n+1}^{(1,0)} - \mathbb{P}_{n+1}^{(0,1)} - \mathbb{P}_{n+1}^{(0,2)})D\mathcal{N}_{n+1}(H_{n+1}^0)f \circ \mathcal{T}_n. \quad (3.6.11)$$

The space of q-independent Hamiltonians is an invariant subspace of the derivative operator. A constant Hamiltonian, and the Hamiltonians $(\hat{\omega}_n \cdot p)$, $(\Omega_n \cdot p)$ and $(\hat{\omega}_n \cdot p)^2$, are eigenvectors with eigenvalue zero. For $\kappa \in \mathbb{N}_0^2$ different from $(0,0)$, $(1,0)$, $(0,1)$ and $(0,2)$, the derivative operator maps the Hamiltonian $(\hat{\omega}_n \cdot p)^{\kappa_1}(\Omega_n \cdot p)^{\kappa_2}$ into

$$D\mathcal{R}_n(H_n^0)(\hat{\omega}_n \cdot p)^{\kappa_1}(\Omega_n \cdot p)^{\kappa_2} = \frac{\|\omega_{n+1}\|^{\kappa_1}}{(\alpha_{n+1}\|T_n^{-1}\Omega_n\|)^{2(\kappa_1-1)+\kappa_2}\|\omega_n\|^{\kappa_1}}(\hat{\omega}_{n+1} \cdot p)^{\kappa_1}(\Omega_{n+1} \cdot p)^{\kappa_2}.$$

Thus, the operator norm of $D\mathcal{R}_n(H_n^0)$ acting on q-independent Hamiltonians can be bounded by

$$\|D\mathcal{R}_n(H_n^0)\mathbb{E}\| \leq \frac{\max\{1, \frac{\|\omega_{n+1}\|}{\|\omega_n\|}\}}{\alpha_{n+1}\|T_n^{-1}\Omega_n\|} \leq 2/3. \quad (3.6.12)$$

The derivative of the n-th step renormalization operator $\mathcal{R}_n$ at a q-independent Hamiltonian $\mathbb{E}H$ is the linear operator $\mathcal{L}_n = D\mathcal{R}_n(\mathbb{E}H) : \mathbb{I}_n^+ \mathcal{A}_n(\rho') \to \mathbb{I}_{n+1}^+ \mathcal{A}_{n+1}(\rho')$, given by its action on an arbitrary $f \in \mathbb{I}_n^+ \mathcal{A}_n(\rho')$,

$$\mathcal{L}_n f = \theta_n/\mu_n D\mathfrak{S}(\mathbb{E}\tilde{H})D\mathcal{N}(\mathbb{E}H'')D\mathcal{V}(\mathbb{E}H')f \circ \mathcal{T}_n. \quad (3.6.13)$$

The operators $\mathfrak{S}$ and $\mathcal{V}$ have been introduced in Proposition 3.4.3 and Proposition 3.4.2, respectively.

The next proposition shows that there is a super-exponential shrinking of the q-dependent modes (which may also depend on the p-variables). In the case of vector fields on a torus, a similar result has been obtained in [57]. In the following, we assume that $H_n = H_n^0 + h_n \in \mathcal{A}_n(\rho')$, $n \in \mathbb{N}_0$, is a sequence of Hamiltonians that form the renormalization orbit of $H_0 \in \overset{+}{\mathbb{I}}_0 \mathcal{A}_0(\rho')$, $\rho' > 0$, componentwise, and that $\mathcal{L}_n$, $n \in \mathbb{N}_0$, is the associated sequence of previously defined operators.

Proposition 3.6.3 *Let $\omega_0 \in \mathcal{D}(\beta)$, for some $\beta \geq 0$. There exist $c_1, c_2 > 0$, such that*

$$\|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})\| \leq c_1^{n-j+1} \frac{\tilde{A}_{n+1}^2 \tilde{A}_n^2}{\tilde{A}_j^2 \tilde{A}_{j-1}^2} e^{-c_2 \Lambda_{j,n}}, \tag{3.6.14}$$

for $n \geq 0$ and $j = 0, \ldots, n$, assuming that $\|h_i\|_{i,\rho'} \leq \zeta_i/2$, for $i = j, \ldots, n$, where, as before, $\zeta_i = C'/(\alpha_i \alpha_{i+1})^2$, $C' > 0$. Here,

$$\Lambda_{j,n}^{2+\beta} = \frac{\tilde{A}_{n+1}}{\max\{\sigma, \varkappa\} \tilde{A}_{j-1}^{1+\beta}}. \tag{3.6.15}$$

Proof: The following properties of the linear operator $\mathcal{L}_n$ will be useful to prove this proposition. First, $\mathcal{L}_n = \overset{+}{\mathbb{I}}_{n+1} \mathcal{L}_n$. Second, when acting on a Fourier mode, this operator changes its index ν into $T_n^\star \nu$, as the derivatives of the elimination, translation and scaling maps do not change the value of ν. We are interested in the action of the operator $\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})$ on the modes indexed by $\nu \neq 0$. A mode with a particular value of $\kappa \in \mathbb{N}_0^2$ can in general produce modes with different κ'-values in $\mathbb{N}_0^2$. We will denote by $\|\kappa'\|_{\min}$ the minimum of the norms of the vectors κ' generated from a mode index κ.

For every $n \in \mathbb{N}_0$, let

$$\begin{aligned} I_n^{1+} &= \{(\nu, \kappa) \in I_n^+ : |\omega_n \cdot \nu| \leq \sigma |\Omega_n \cdot \nu|\}, \\ I_n^{2+} &= \{(\nu, \kappa) \in I_n^+ : |\omega_n \cdot \nu| \leq \varkappa \|\kappa\|\}, \end{aligned}$$

be the two subsets of I_n^+. We also define the following subset of I_j^{1+}, for $j = 0, \dots, n$,

$$V_{j,n}^+ = \{(\nu, \kappa) \in I_j^{1+} : (T_n^\star \dots T_j^\star \nu, \kappa') \in I_{n+1}^{1+}\}.$$

For every $(\nu, \kappa) \in V_{j,n}^+$, ν must satisfy the resonant condition

$$|\omega_{n+1} \cdot T_n^\star \dots T_j^\star \nu| \le \sigma |\Omega_{n+1} \cdot T_n^\star \dots T_j^\star \nu|. \tag{3.6.16}$$

Notice that $\omega_{n+1} \cdot T_n^\star \dots T_j^\star \nu = T_j \dots T_n \omega_{n+1} \cdot \nu$. As $T_n^{-1}\omega_n = \alpha_{n+1}^{-1}\omega_{n+1}$, we obtain

$$T_j \dots T_n \omega_{n+1} = \omega_j \prod_{i=j+1}^{n+1} \alpha_i. \tag{3.6.17}$$

Similarly, $\Omega_{n+1} \cdot T_n^\star \dots T_j^\star \nu = T_j \dots T_n \Omega_{n+1} \cdot \nu$. From $T_n^{-1}\Omega_n = \Omega_{n+1}\|T_n^{-1}\Omega_n\|$, we find

$$T_j \dots T_n \Omega_{n+1} = \Omega_j \prod_{i=j}^{n} \frac{1}{\|T_i^{-1}\Omega_i\|}.$$

The condition (3.6.16) can then be written in the form

$$\left(\frac{1}{\sigma} \prod_{i=j+1}^{n+1} \alpha_i \|T_{i-1}^{-1}\Omega_{i-1}\| + \frac{1}{\|\omega_j\|} \right) |\omega_j \cdot \nu| \le |\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|. \tag{3.6.18}$$

A lower bound on $|\omega_j \cdot \nu|$ can be obtained by using the Diophantine property of ω_0, i.e. that there exists $C_0 > 0$, such that $|\omega_0 \cdot \nu| \ge C_0 \|\nu\|^{-(1+\beta)}$, for any $\nu \in \mathbb{Z}^2$. This property implies that

$$|\omega_j \cdot \nu| = |\omega_0 \cdot {T_0^\star}^{-1} \dots T_{j-1}^{\star -1} \nu| \prod_{i=1}^{j} \alpha_i \ge \frac{C_0 \prod_{i=1}^{j} \alpha_i}{\|{T_0^\star}^{-1} \dots T_{j-1}^{\star -1}\nu\|^{1+\beta}},$$

for $j \ge 1$. As $\|{T_0^\star}^{-1} \dots T_{j-1}^{\star -1}\nu\| = \|P_{j-1}^{-1}\nu\| \le (p_{j-1} + q_{j-1})\|\nu\| \le (1 + 1/\alpha_0)\tilde{A}_{j-1}\|\nu\|$

and $\|\nu\| \leq 2(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)$, there exists $c > 1$, such that

$$|\omega_j \cdot \nu| \geq \frac{C_0 \tilde{A}_j}{\alpha_0 (2c\tilde{A}_{j-1})^{1+\beta} (|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)^{1+\beta}} , \tag{3.6.19}$$

for $j \geq 0$. The bounds (3.6.18) and (3.6.19) imply that

$$|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu| \geq \frac{\left(\frac{1}{\sigma} \prod_{i=j+1}^{n+1} \alpha_i \|T_{i-1}^{-1} \Omega_{i-1}\| + \frac{1}{\|\omega_j\|}\right) C_0 \tilde{A}_j}{\alpha_0 (2c\tilde{A}_{j-1})^{1+\beta} (|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)^{1+\beta}} ,$$

and, thus,

$$(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)^{2+\beta} \geq \frac{C_0}{\alpha_0 (2c)^{1+\beta} \sigma} \frac{\tilde{A}_{n+1} \prod_{i=j+1}^{n+1} \|T_{i-1}^{-1} \Omega_{i-1}\|}{\tilde{A}_{j-1}^{1+\beta}} .$$

Using the bounds obtained in Proposition 3.6.2, we find that there exists $c' > 0$, such that

$$(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|) \geq c' \Lambda'_{j,n} , \tag{3.6.20}$$

where

$$\Lambda'^{2+\beta}_{j,n} = \frac{\tilde{A}_{n+1} \tilde{A}_n}{\sigma \tilde{A}_{j-1}^{1+\beta} \tilde{A}_{j-1}} . \tag{3.6.21}$$

Now consider the modes indexed by $(\nu, \kappa) \in \mathbb{Z}^2 \times \mathbb{N}_0^2$ that belong to $I_j^+ \backslash V_{j,n}^+$. Let $(T_n^\star \ldots T_j^\star \nu, \kappa') \in I_{n+1}^{2+}$ be the index of a mode generated by the action of the operator $\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j (\mathbb{I} - \mathbb{E})$ on such a mode. The following condition must be satisfied,

$$|\omega_{n+1} \cdot T_n^\star \ldots T_j^\star \nu| \leq \varkappa \|\kappa'\|_{\min} \leq \varkappa \|\kappa'\| . \tag{3.6.22}$$

From the inequality (3.6.22), using the identity (3.6.17) and the Diophantine bound (3.6.19), we find that

$$\|\kappa'\| \geq \|\kappa'\|_{\min} \geq \frac{C_0}{\varkappa\alpha_0(2c)^{1+\beta}} \frac{\tilde{A}_{n+1}}{\tilde{A}_{j-1}^{1+\beta}(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)^{1+\beta}} .$$

Define $W_{j,n}^+ = \{(\nu, \kappa) \in I_j^+ : (T_n^\star \dots T_j^\star \nu, \kappa') \in I_{n+1}^{2+},\ |\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu| \geq \|\kappa'\|_{\min}\}$. If $(\nu, \kappa) \in (I_j^+ \backslash V_{j,n}^+) \cap W_{j,n}^+$, then

$$(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|) \geq c'' \Lambda''_{j,n} , \tag{3.6.23}$$

where

$$\Lambda_{j,n}^{\prime\prime 2+\beta} = \frac{\tilde{A}_{n+1}}{\varkappa \tilde{A}_{j-1}^{1+\beta}} , \tag{3.6.24}$$

and $c'' > 0$. If $(T_n^\star \dots T_j^\star \nu, \kappa') \in I_{n+1}^{2+}$ and $(\nu, \kappa) \in (I_j^+ \backslash V_{j,n}^+) \cap (I_j^+ \backslash W_{j,n}^+)$, then

$$\|\kappa'\|_{\min} \geq c'' \Lambda''_{j,n} . \tag{3.6.25}$$

Let $\mathbb{V}_{j,n}^+ : \mathcal{A}_j(\rho') \to \mathcal{A}_j(\rho')$ be the projection operator on $\mathcal{A}_j(\rho')$ over the indices in $V_{j,n}^+$, defined by the action

$$\mathbb{V}_{j,n}^+ f = \sum_{(\nu,\kappa) \in V_{j,n}^+} f_{\nu,\kappa} (\hat{\omega} \cdot p)^{\kappa_1} (\Omega \cdot p)^{\kappa_2} e^{iq\cdot\nu} ,$$

on an arbitrary $f \in \mathcal{A}_j(\rho')$. Let $\bar{\mathbb{V}}_{j,n}^+ : \mathcal{A}_j(\rho') \to \mathcal{A}_j(\rho'')$ be the same projection followed by an analytic inclusion in the q variables, obtained by restricting the domain of $f \in \mathcal{A}_j(\rho')$ to $D_j(\rho'')$, where $\rho'_1 > \rho''_1 > 0$ and $\rho'_2 = \rho''_2$. As

$$\|\bar{\mathbb{V}}_{j,n}^+ f\|_{j,\rho''} = \sum_{(\nu,\kappa) \in V_{j,n}^+} |f_{\nu,\kappa}| \rho_2^{\prime\|\kappa\|} e^{\rho'_1(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)} e^{-(\rho'_1 - \rho''_1)(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)} ,$$

we have $\|\bar{\mathbb{V}}^+_{j,n}f\|_{j,\rho''} \leq \|\bar{\mathbb{V}}^+_{j,n}\|\|f\|_{j,\rho'}$, with $\|\bar{\mathbb{V}}^+_{j,n}\| \leq e^{-(\rho'_1-\rho''_1)c'\Lambda'_{j,n}}$, as follows from the bound (3.6.20).

Similarly, let $\mathbb{W}^+_{j,n} : \mathcal{A}_j(\rho') \to \mathcal{A}_j(\rho')$, be the projection operator on $\mathcal{A}_j(\rho')$ over the indexes in $W^+_{j,n}$ and $\bar{\mathbb{W}}^+_{j,n} : \mathcal{A}_j(\rho') \to \mathcal{A}_j(\rho'')$ the same projection followed by an analytic inclusion in the q variables. Here, as before, $\rho'_1 > \rho''_1 > 0$ and $\rho'_2 = \rho''_2$. Since, for $f \in \mathcal{A}_j(\rho')$,

$$\|\bar{\mathbb{W}}^+_{j,n}f\|_{j,\rho''} = \sum_{(\nu,\kappa)\in W^+_{j,n}} |f_{\nu,\kappa}|\rho_2'^{\|\kappa\|} e^{\rho'_1(|\hat{\omega}_j\cdot\nu|+|\Omega_j\cdot\nu|)} e^{-(\rho'_1-\rho''_1)(|\hat{\omega}_j\cdot\nu|+|\Omega_j\cdot\nu|)},$$

from the bound (3.6.23), we find that the operator norm of $\bar{\mathbb{W}}^+_{j,n}$ satisfies $\|\bar{\mathbb{W}}^+_{j,n}\| \leq e^{-(\rho'_1-\rho''_1)c''\Lambda''_{j,n}}$.

Let $\bar{\mathbb{I}}_n : \mathcal{A}_n(\varrho'') \to \mathcal{A}_n(\rho')$, be the inclusion map obtained by restricting the domain of the functions $f \in \mathcal{A}_n(\varrho'')$ to $D_n(\rho')$, where $\varrho''_2 > \rho'_2 > 0$ and $\varrho''_1 = \rho'_1$. As

$$\|\bar{\mathbb{I}}_n f\|_{n,\rho'} = \sum_{(\nu,\kappa')\in I_n} |f_{\nu,\kappa'}|\varrho_2''^{\|\kappa'\|} e^{\varrho''_1(|\hat{\omega}_n\cdot\nu|+|\Omega_n\cdot\nu|)} e^{-(\ln\varrho''_2-\ln\rho'_2)\|\kappa'\|} \leq \|\bar{\mathbb{I}}_n\|\|f\|_{n,\varrho''},$$

the norm of the inclusion map $\bar{\mathbb{I}}_n$ acting on functions that are composed only of modes with $\|\kappa'\| \geq \|\kappa'\|_{\min}$ satisfying the inequality (3.6.25), can be bounded by $\|\bar{\mathbb{I}}_n\| \leq e^{-(\ln\varrho''_2-\ln\rho'_2)c''\Lambda''_{j,n}}$.

To obtain the desired bound (3.6.14), we write the operator $\mathcal{L}_n\circ\cdots\circ\mathcal{L}_j(\mathbb{I}-\mathbb{E})$ as

$$\begin{aligned}\mathcal{L}_n\circ\cdots\circ\mathcal{L}_j(\mathbb{I}-\mathbb{E}) =& \mathcal{L}_n\circ\cdots\circ\mathcal{L}_{j+1}(\mathbb{I}-\mathbb{E})\overline{\mathcal{L}}^{(1)}_j\bar{\mathbb{V}}^+_{j,n} \\ &+ \mathcal{L}_n\circ\cdots\circ\mathcal{L}_{j+1}(\mathbb{I}-\mathbb{E})\overline{\mathcal{L}}^{(1)}_j\bar{\mathbb{W}}^+_{j,n}(\mathbb{I}-\mathbb{V}^+_{j,n}) \\ &+ \bar{\mathbb{I}}_{n+1}\overline{\mathcal{L}}^{(2)}_n\circ\mathcal{L}_{n-1}\circ\cdots\circ\mathcal{L}_j(\mathbb{I}-\mathbb{E})(\mathbb{I}-\mathbb{W}^+_{j,n})(\mathbb{I}-\mathbb{V}^+_{j,n}).\end{aligned} \tag{3.6.26}$$

Here, $\overline{\mathcal{L}}_n^{(1)} : \mathbb{I}_n^+ \mathcal{A}_n(\rho'') \to \mathbb{I}_{n+1}^+ \mathcal{A}_{n+1}(\rho')$ and $\overline{\mathcal{L}}_n^{(2)} : \mathbb{I}_n^+ \mathcal{A}_n(\rho') \to \mathbb{I}_{n+1}^+ \mathcal{A}_{n+1}(\varrho'')$, $n \in \mathbb{N}_0$, are the derivatives of the n-th step renormalization operator at $\mathbb{E}H_n$. With superscripts (1) and (2), we explicitly emphasize that these operators actually improve analyticity in q and p variables, respectively. For some $\rho'', \varrho'' > 0$ satisfying $0 < \rho_1'' < \rho_1'$, $\rho_2'' = \rho_2'$, $\varrho_1'' = \rho_1'$ and $\varrho_2'' > \rho_2'$, the construction of the analyticity improving operators is possible, as mentioned in Remark 3.4.5.

The norm $\|D\mathcal{R}_i(H_i^0)\|$, $i \in \mathbb{N}_0$, can be bounded by a constant times θ_i/μ_i. By Cauchy's estimate, we have

$$\|D\mathcal{R}_i(H_i) - D\mathcal{R}_i(H_i^0)\| \leq \frac{2\|h_i\|_{i,\rho'}}{(\zeta_i - \|h_i\|_{i,\rho'})^2}\,, \tag{3.6.27}$$

where $D\mathcal{R}_i(H_i)$ is the derivative of the i^{th}-step renormalization operator $\mathcal{R}_i$ at a Hamiltonian $H_i = H_i^0 + h_i$. In particular, the inequality (3.6.27) is satisfied by $\mathcal{L}_i$, the derivative of the i^{th}-step renormalization operator at $\mathbb{E}H_i$. As, by assumption, $\|h_i\|_{i,\rho'} \leq \zeta_i/2$, we have the bound $\|\mathcal{L}_i\| \leq c_3\theta_i/\mu_i$, with $c_3 > 0$.

Using the bounds on the operator norms $\|\bar{\mathbb{V}}_{j,n}^+\|, \|\bar{\mathbb{W}}_{j,n}^+\|$ and $\|\bar{\mathbb{I}}_n\|$, obtained above, we find that

$$\|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \leq c_3 e^{-c_2\Lambda_{j,n}} \left(2\frac{\theta_j}{\mu_j} \|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_{j+1}(\mathbb{I}-\mathbb{E})\| + \frac{\theta_n}{\mu_n}\|\mathcal{L}_{n-1} \circ \cdots \circ \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \right),$$

where $\Lambda_{j,n} \leq \min\{\Lambda_{j,n}', \Lambda_{j,n}''\}$, with $\Lambda_{j,n}'$ and $\Lambda_{j,n}''$ given by the expressions (3.6.21) and (3.6.24), respectively, and $c_2 = \min\{(\rho_1' - \rho_1'')c', (\rho_1' - \rho_1'')c'', (\ln\varrho_2'' - \ln\rho_2')c''\}$.

Using the bounds on the norms of $\mathcal{L}_i$, for $i = j, \ldots, n$, we obtain

$$\|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \leq 3c_3^{n-j+1} e^{-c_2\Lambda_{j,n}} \prod_{i=j}^{n} \frac{\theta_i}{\mu_i}\,.$$

The bound (3.6.14) follows from this inequality and the bounds in Proposition 3.6.2.

QED

After the first $(n+1)$ renormalization steps the perturbation can be separated into two parts,

$$h_{n+1} = \mathbb{E}h_{n+1} + (\mathbb{I} - \mathbb{E})h_{n+1}\,. \tag{3.6.28}$$

The q-independent part of the perturbation can be determined by

$$\mathbb{E}h_{n+1} = D\mathcal{R}_n(H_n^0)\mathbb{E}h_n + \mathbb{E}\mathcal{O}_n^0(\|h_n\|^2_{n,\rho'})\,, \tag{3.6.29}$$

or after applying this equality recursively,

$$\begin{aligned}\mathbb{E}h_{n+1} &= D\mathcal{R}_n(H_n^0)\ldots D\mathcal{R}_0(H_0^0)\mathbb{E}h_0 \\ &\quad + \sum_{j=1}^{n} D\mathcal{R}_n(H_n^0)\ldots D\mathcal{R}_j(H_j^0)\mathbb{E}\mathcal{O}_{j-1}^0(\|h_{j-1}\|^2_{j-1,\rho'}) + \mathbb{E}\mathcal{O}_n^0(\|h_n\|^2_{n,\rho'})\,.\end{aligned} \tag{3.6.30}$$

Here $\mathcal{O}_n^0(\|h_n\|^2_{n,\rho'})$ denotes the second-order remainder of the Taylor expansion of $\mathcal{R}_n(H_n)$ about H_n^0. The norm of $\mathcal{O}_n^0(\|h_n\|^2_{n,\rho'})$ will be denoted by F_n^2 and can be estimated by the bound (3.4.4).

In order to estimate the q-dependent part of the perturbation, we perform the Taylor expansion of $\mathcal{R}_n(H_n)$ about $\mathbb{E}H_n$, the q-independent part of the Hamiltonian H_n. The q-dependent part of the perturbation is

$$(\mathbb{I} - \mathbb{E})h_{n+1} = \mathcal{L}_n(\mathbb{I} - \mathbb{E})h_n + (\mathbb{I} - \mathbb{E})\mathcal{O}_n(\|(\mathbb{I} - \mathbb{E})h_n\|^2_{n,\rho'})\,, \tag{3.6.31}$$

where $\mathcal{O}_n(\|(\mathbb{I} - \mathbb{E})h_n\|^2_{n,\rho'})$ denotes the second-order remainder of the Taylor expansion of $\mathcal{R}_n(H_n)$ about $\mathbb{E}H_n$. The norm of this remainder is of the order of $\|(\mathbb{I} - \mathbb{E})h_n\|^2_{n,\rho'}$ and will be denoted by G_n^2.

After successive applications of the previous recursion relation, we obtain

$$(\mathbb{I}-\mathbb{E})h_{n+1} = \mathcal{L}_n \dots \mathcal{L}_0(\mathbb{I}-\mathbb{E})h_0 + \sum_{j=1}^{n} \mathcal{L}_n \dots \mathcal{L}_j(\mathbb{I}-\mathbb{E})\mathcal{O}_{j-1}(\|(\mathbb{I}-\mathbb{E})h_{j-1}\|^2_{j-1,\rho'})$$
$$+ (\mathbb{I}-\mathbb{E})\mathcal{O}_n(\|(\mathbb{I}-\mathbb{E})h_n\|^2_{n,\rho'}) \,. \quad (3.6.32)$$

We will use this identity to estimate the decrease of the norm of the q-dependent part of the perturbation in the following theorem.

Theorem 3.6.4 *Let $\omega_0 \in \mathcal{D}(\beta)$, $0 \le \beta < \sqrt{2}-1$, and let $\rho'_1 > 0$. There exist $\tau > 2$ and $\rho'_2, \sigma, \varkappa, C > 0$, such that if $H_0 = H_0^0 + h_0 \in \overset{+}{\mathbb{I}}_0 \mathcal{A}_0(\rho')$ and $\|h_0\|_{0,\rho'} \le C^2 < 1$, then $\|(\mathbb{I}-\mathbb{E})h_n\|_{n,\rho'} \le C\tilde{A}_n^{-\tau} < \zeta_n/2$, $n \ge 0$, where $\zeta_n = C'/(\alpha_n\alpha_{n+1})^2$, $C' > 0$. Furthermore, if $\beta < (\sqrt{161}-11)/10$, then $\|h_n\|_{n,\rho'} \le C\tilde{A}_{n-1}^{-2} < \zeta_n/2$.*

Proof: We find first, using the Diophantine bound (3.6.9), that if $\beta \le \sqrt{2}-1$ and $\tau > 2$, there exists $C > 0$, such that

$$\frac{\zeta_n}{2} = \frac{C'}{2\alpha_n^2\alpha_{n+1}^2} \ge \frac{C'}{2\tilde{K}^4\tilde{A}_{n-1}^{2\beta}\tilde{A}_n^{2\beta}} \ge \frac{C'}{2\tilde{K}^{4+2\beta}\tilde{A}_{n-1}^{2\beta}\tilde{A}_{n-1}^{2\beta(1+\beta)}} > \frac{C}{\tilde{A}_{n-1}^2} > \frac{C}{\tilde{A}_n^\tau} \,.$$

Let $\tau' = \ln c_1/\ln\gamma$, where $c_1 > 1$ is the constant from Proposition 3.6.3. As $\tilde{A}_n/\tilde{A}_{j-1} \ge \gamma^{n-j}$, $0 \le j \le n$, the inequality (3.6.14) implies

$$\|\mathcal{L}_n \circ \dots \circ \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \le \frac{\tilde{A}_{n+1}^{4+\tau'}}{\tilde{A}_{j-1}^{4+\tau'}} e^{-c_2\Lambda_{j,n}} \,. \quad (3.6.33)$$

Using the inequality $e^{-t} \le (s/t)^s$, valid for any $t > 0$ and $s > 0$, we find that, for any $\tau'' > 0$,

$$\frac{\tilde{A}_{n+1}^{4+\tau'}}{\tilde{A}_{j-1}^{4+\tau'}} e^{-c_2\Lambda_{j,n}} \le (\max\{\sigma,\varkappa\})^{\tau''} \left(\frac{\tau''(2+\beta)}{c_2}\right)^{\tau''(2+\beta)} \frac{\tilde{A}_{j-1}^{\tau''(1+\beta)-(4+\tau')}}{\tilde{A}_{n+1}^{\tau''-(4+\tau')}} \,.$$

Let $\tau'' > 0$ be given, such that $\tau = \tau'' - \tau' - 4 > 2$. There exist $\sigma, \varkappa > 0$, such that

$$\frac{\tilde{A}_{n+1}^{4+\tau'}}{\tilde{A}_{j-1}^{4+\tau'}} e^{-c_2 \Lambda_{j,n}} \le (\max\{\sigma, \varkappa\})^{\tau''} \left(\frac{\tau''(2+\beta)}{c_2}\right)^{\tau''(2+\beta)} \frac{\tilde{A}_{j-1}^{\tau(1+\beta)+(4+\tau')\beta}}{\tilde{A}_{n+1}^{\tau}} \le \frac{\tilde{A}_{j-1}^{\tau(1+\beta)+(4+\tau')\beta}}{6\tilde{A}_{n+1}^{\tau}},$$

and thus,

$$\|\mathcal{L}_n \circ \cdots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})\| \le \frac{\tilde{A}_{j-1}^{\tau(1+\beta)+(4+\tau')\beta}}{6\tilde{A}_{n+1}^{\tau}}. \tag{3.6.34}$$

We will prove the claim by induction. There exists $0 < C < 1$, such that

$$\|(\mathbb{I} - \mathbb{E})h_0\|_{0,\rho'} \le \|h_0\|_{0,\rho'} \le C^2 < C\tilde{A}_0^{-\tau} < C < \zeta_0/2. \tag{3.6.35}$$

Therefore, for $n = 0$, the claim is true. Assume that that the claim holds for $0 < j \le n$. Thus, there exists $C > 0$, such that $\|(\mathbb{I} - \mathbb{E})h_j\|_{j,\rho'} \le C\tilde{A}_j^{-\tau} < \zeta_j/2$ and $\|h_j\|_{j,\rho'} \le C\tilde{A}_{j-1}^{-2} < \zeta_j/2$. We will show that the claim is true for $j = n+1$.

Using the identity (3.6.32), we obtain

$$\|(\mathbb{I}-\mathbb{E})h_{n+1}\|_{n+1,\rho'} \le \|\mathcal{L}_n \ldots \mathcal{L}_0(\mathbb{I}-\mathbb{E})h_0\|_{n+1,\rho'} + \sum_{j=1}^{n} \|\mathcal{L}_n \ldots \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \cdot G_{j-1}^2 + G_n^2. \tag{3.6.36}$$

We will estimate the size of the terms on the right hand side of the inequality (3.6.36).

As $\|h_0\|_{0,\rho'} \le C^2 < 1$, the inequality (3.6.34), for $j = 0$, implies a bound on the first term

$$\|\mathcal{L}_n \ldots \mathcal{L}_0(\mathbb{I} - \mathbb{E})h_0\|_{n+1,\rho'} \le \frac{C}{6\tilde{A}_{n+1}^{\tau}}. \tag{3.6.37}$$

Using Cauchy's formula, one can obtain an estimate on the norm of the second-order remainder $\mathcal{O}_n(\|(\mathbb{I}-\mathbb{E})h_n\|^2_{n,\rho'})$, analogous to the bound (3.4.4),

$$G_n^2 \le \frac{\|(\mathbb{I}-\mathbb{E})h_n\|^2_{n,\rho'}}{(\zeta_n - \|\mathbb{E}h_n\|_{n,\rho'})(\zeta_n - \|\mathbb{E}h_n\|_{n,\rho'} - \|(\mathbb{I}-\mathbb{E})h_n\|_{n,\rho'})}\,.$$

Applying the inductive hypothesis and the Diophantine condition in the form (3.6.9), or equivalently, $\alpha_{n+1} \le \tilde{K}\tilde{A}_n^{\beta}$, we further obtain that, for some $C > 0$,

$$G_n^2 \le \frac{4C^2}{\zeta_n^2 \tilde{A}_n^{2\tau}} \le \frac{4C^2 \alpha_n^4 \alpha_{n+1}^4}{C'^2 \tilde{A}_n^{2\tau}} \le \frac{4C^2 \tilde{K}^{2\tau/(1+\beta)+8} \tilde{A}_n^{4\beta} \tilde{A}_{n-1}^{4\beta}}{C'^2 \tilde{A}_{n+1}^{2\tau/(1+\beta)}} < \frac{C}{6\tilde{A}_{n+1}^{\tau}}\,, \qquad (3.6.38)$$

if $0 \le \beta < 1$ and $\tau \ge 8\beta(1+\beta)/(1-\beta)$. Using the bounds (3.6.34) and (3.6.38), we can estimate the sum

$$\begin{aligned}\sum_{j=1}^{n} \|\mathcal{L}_n \dots \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \cdot G_{j-1}^2 &\le \frac{2C^2}{3\tilde{A}_{n+1}^{\tau}} \sum_{j=1}^{n} \frac{\tilde{A}_{j-1}^{\tau(1+\beta)+(4+\tau')\beta}}{\zeta_{j-1}^2 \tilde{A}_{j-1}^{2\tau}} \\ &\le \frac{2C^2\tilde{K}^8}{3C'^2\tilde{A}_{n+1}^{\tau}} \sum_{j=1}^{n} \frac{1}{\tilde{A}_{j-1}^{\tau(1-\beta)-(12+\tau')\beta}}\,.\end{aligned}$$

Since for $\alpha_0 > 1$, we have $\tilde{A}_{j-1} \ge \gamma^{j-1}$, the previous sum can be bounded by some positive constant, if $0 \le \beta < 1$ and $\tau > \beta(\tau'+12)/(1-\beta)$. Thus, there exists $C > 0$, such that

$$\sum_{j=1}^{n} \|\mathcal{L}_n \dots \mathcal{L}_j(\mathbb{I}-\mathbb{E})\| \cdot G_{j-1}^2 \le \frac{C}{6\tilde{A}_{n+1}^{\tau}}\,. \qquad (3.6.39)$$

Finally, if $0 \le \beta < 1$ is given, and constants $\tau > 2$ and $\sigma, \varkappa, C > 0$ are chosen according to the various conditions stated above, using the bounds (3.6.37), (3.6.38) and (3.6.39), we obtain

$$\|(\mathbb{I}-\mathbb{E})h_{n+1}\|_{n+1,\rho'} \le \frac{C}{6\tilde{A}_{n+1}^{\tau}} + \frac{C}{6\tilde{A}_{n+1}^{\tau}} + \frac{C}{6\tilde{A}_{n+1}^{\tau}} = \frac{C}{2\tilde{A}_{n+1}^{\tau}} < \frac{C}{\tilde{A}_{n+1}^{\tau}} < \frac{\zeta_{n+1}}{2}\,. \qquad (3.6.40)$$

Concerning the q-independent part of the perturbation, from the identity (3.6.30), we find that

$$\|\mathbb{E}h_{n+1}\|_{n+1,\rho'} \leq \|D\mathcal{R}_n(H_n^0)\dots D\mathcal{R}_0(H_0^0)\mathbb{E}h_0\|_{n+1,\rho'} + \sum_{j=1}^{n}\|D\mathcal{R}_n(H_n^0)\dots D\mathcal{R}_j(H_j^0)\mathbb{E}\|\cdot F_{j-1}^2 + F_n^2\,. \quad (3.6.41)$$

Using the bound (3.6.12) on the derivative of a one-step renormalization operator acting on q-independent Hamiltonians and the first of the inequalities (3.6.10), one obtains that there exists $C>0$, such that

$$\|D\mathcal{R}_n(H_n^0)\dots D\mathcal{R}_0(H_0^0)\mathbb{E}h_0\|_{n,\rho'} \leq \frac{4\alpha_0^2}{(1+\alpha_0)\tilde{A}_n^2}\|h_0\|_{0,\rho'} \leq \frac{4\alpha_0^2}{(1+\alpha_0)\tilde{A}_n^2}C^2 < \frac{C}{6\tilde{A}_n^2}\,. \quad (3.6.42)$$

From the estimate (3.4.4) obtained in Theorem 3.4.4, we have the following bound on the norm of the second-order remainder, $F_n^2 \leq \zeta_n^{-1}(\zeta_n - \|h_n\|_{n,\rho'})^{-1}\|h_n\|_{n,\rho'}^2$. Using the inductive hypothesis and the Diophantine property of ω_0, we find that there exists $C>0$, such that

$$F_j^2 \leq \frac{2C^2}{\zeta_j^2\tilde{A}_{j-1}^4} \leq \frac{2C^2\tilde{K}^{4/(1+\beta)}\alpha_j^4\alpha_{j+1}^4}{C'^2\tilde{A}_j^{4/(1+\beta)}} \leq \frac{2C^2\tilde{K}^{4/(1+\beta)+8}\tilde{A}_{j-1}^{4\beta}\tilde{A}_j^{4\beta}}{C'^2\tilde{A}_j^{4/(1+\beta)}} < \frac{C}{6\tilde{A}_j^2}\,, \quad (3.6.43)$$

for $0\leq\beta<(\sqrt{41}-5)/8$. Since, from the bound (3.6.12) and the inequalities (3.6.10), one has

$$\|D\mathcal{R}_n(H_n^0)\dots D\mathcal{R}_j(H_j^0)\mathbb{E}\| \leq \frac{8\alpha_0\tilde{A}_j\tilde{A}_{j-1}}{(1+\alpha_0)\tilde{A}_n^2}\,,$$

the sum on the right hand side of the inequality (3.6.41) can be estimated by

$$\begin{aligned}\sum_{j=1}^{n}\|D\mathcal{R}_n(H_n^0)\dots D\mathcal{R}_j(H_j^0)\mathbb{E}\|\cdot F_{j-1}^2 &\leq \frac{16\alpha_0 C^2\tilde{K}^8}{(1+\alpha_0)C'^2\tilde{A}_n^2}\sum_{j=1}^{n}\frac{\tilde{A}_j\tilde{A}_{j-1}^{1+4\beta}\tilde{A}_{j-2}^{4\beta}}{\tilde{A}_{j-2}^4}\\ &\leq \frac{16\alpha_0 C^2\tilde{K}^{11+5\beta}}{(1+\alpha_0)C'^2\tilde{A}_n^2}\sum_{j=1}^{n}\frac{1}{\tilde{A}_{j-2}^{4-(2+5\beta)(1+\beta)-4\beta}}\,.\end{aligned}$$

If $2 - 11\beta - 5\beta^2 > 0$, the sum on the right side can be bounded by a positive constant, as $\tilde{A}_j$ grows at least exponentially with j. Thus, there exists $C > 0$, such that

$$\sum_{j=1}^{n} \|D\mathcal{R}_n(H_n^0) \dots D\mathcal{R}_j(H_j^{(0)})\mathbb{E}\| \cdot F_{j-1}^2 \leq \frac{C}{6\tilde{A}_n^2}\,. \tag{3.6.44}$$

Using the bounds (3.6.42), (3.6.43) and (3.6.44), the inequality (3.6.41) implies

$$\|\mathbb{E}h_{n+1}\|_{n+1,\rho'} \leq \frac{C}{6\tilde{A}_n^2} + \frac{C}{6\tilde{A}_n^2} + \frac{C}{6\tilde{A}_n^2} = \frac{C}{2\tilde{A}_n^2}\,. \tag{3.6.45}$$

Finally, taking into account the estimates (3.6.40) and (3.6.45), we find that

$$\|h_{n+1}\|_{n+1,\rho'} = \|(\mathbb{I}-\mathbb{E})h_{n+1}\|_{n+1,\rho'} + \|\mathbb{E}h_{n+1}\|_{n+1,\rho'} \leq \frac{C}{2\tilde{A}_{n+1}^\tau} + \frac{C}{2\tilde{A}_n^2} \leq \frac{C}{\tilde{A}_n^2} < \frac{\zeta_{n+1}}{2}\,.$$

This completes the proof of the claim. **QED**

Corollary 3.6.5 *Let $\omega_0 \in \mathcal{D}(\beta)$, with $0 \leq \beta < (\sqrt{161} - 11)/10$. Given $\rho_1' > 0$, there exist $\rho_2', \sigma, \varkappa, \tilde{C} > 0$ and a non-empty open neighborhood $B_{0,\rho'}^+ \subset \mathbb{I}_0^+ \mathcal{A}_0(\rho')$ of H_0^0, such that for all Hamiltonians $H_0 \in B_{0,\rho'}^+$ and $n \in \mathbb{N}_0$, $\|H_n - H_n^0\|_{n,\rho'} \leq \tilde{C}\gamma^{-2n}$.*

Proof: The proof of this Corollary follows directly from Theorem 3.6.4 and the fact that $\tilde{A}_n$ grows at least exponentially with $n \geq 0$, i.e. $\tilde{A}_n \geq \gamma^n$, for $\alpha_0 > 1$. **QED**

Remark 3.6.6 *In the context of Remark 3.4.6, the set of Hamiltonians in $\mathbb{I}_0^+ \mathcal{A}_0(\rho')$ that satisfies the assumptions of Theorem 3.6.4 and the neighborhood $B_{0,\rho'}^+$ in Corollary 3.6.5 could be replaced with an open ball $B_{0,\rho'} \subset \mathcal{A}_0(\rho')$.*

3.7 A proof of a KAM theorem

In this section, we apply the result about the convergence of the constructed renormalization scheme for $\omega_0 \in \mathcal{D}(\beta)$, with $0 \leq \beta < (\sqrt{161} - 11)/10$, to prove a KAM theorem for near-integrable Hamiltonians which are degenerate in the Kolmogorov sense. We construct the invariant tori with Diophantine frequency vectors for Hamiltonians approaching the trivial limit set under the renormalization.

3.7.1 Definition of invariant tori and formal identities

We start by providing some definitions and some formal identities that will be used in the construction of invariant tori.

Given $\delta_1 > 0$, we define

$$D_{n,0}(\delta_1) = \{q \in \mathbb{C}^2 : |\mathrm{Im}\, \omega'_n \cdot q| < \delta_1,\ |\mathrm{Im}\, \Omega'_n \cdot q| = 0\} \times \{0\}\,. \tag{3.7.1}$$

Definition 3.7.1 *Given $r > 0$ and $\delta_1 > 0$, let $\mathcal{A}_{n,0}(\delta_1)$ be the space of functions f, analytic on $D_{n,0}(\delta_1)$, 2π-periodic in both q-variables, with the finite norm*

$$\|f\|_{n,\delta_1} = \sum_{\nu \in \mathbb{Z}^2} |f_\nu| (1 + |\Omega_n \cdot \nu|)^r e^{\delta_1 |\hat{\omega}_n \cdot \nu|}\,, \tag{3.7.2}$$

where f_ν are the Fourier coefficients of $f = \sum_{\nu \in \mathbb{Z}^2} f_\nu e^{iq\cdot\nu}$.

Let Φ_n be the flow for the vector field X_{H_n} generated by the Hamiltonian H_n and Ψ_n is the flow for the vector field $K_n = (\omega_n, 0)$, generated by the Hamiltonian $H_n^K = \omega_n \cdot p$, i.e. $\Psi_n^s(q, 0) = (q + \omega_n s, 0)$, with $s \in \mathbb{R}$.

Definition 3.7.2 *We say that $H_n \in \mathcal{A}_n(\rho)$ has an invariant torus with frequency vector ω_n if there exists a continuous map $\Gamma_n : D_{n,0}(\delta_1) \to D_n(\rho)$, with $\rho > 0$,* *componentwise, and a continuous function* $t : \mathbb{R} \to \mathbb{R}$*, such that for all* $s \in \mathbb{R}$*,*

$$\Phi_n^{t(s)} \circ \Gamma_n = \Gamma_n \circ \Psi_n^s. \tag{3.7.3}$$

Notice that an invariant torus of a Hamiltonian is defined as the conjugacy between the Hamiltonian flow and a linear flow of an integrable Hamiltonian.

The flow Φ_n for the Hamiltonian H_n and the flow Φ_{n+1} for the renormalized Hamiltonian $H_{n+1} = \mathcal{R}_n(H_n)$ can be related by

$$\Lambda_n \circ \Phi_{n+1}^t = \Phi_n^{\theta'_n t} \circ \Lambda_n. \tag{3.7.4}$$

This identity is valid for $t \in \mathbb{C}$ on any domain where the compositions are well-defined. The identity formally follows from

$$\frac{d}{dt} f \circ \Phi_n^{\theta'_n t} \circ \Lambda_n \;|_{t=0} = \theta'_n \{f, H_n\} \circ \Lambda_n = \frac{\theta'_n}{\mu'_n} \{f \circ \Lambda_n, H_n \circ \Lambda_n\} = \{f \circ \Lambda_n, \mathcal{R}_n(H_n)\},$$

where we have used the identity ${\mu'_n}^{-1}\{f \circ \Lambda_n, g \circ \Lambda_n\} = \{f, g\} \circ \Lambda_n$, for complex valued functions f and g defined on a neighborhood of $\mathbb{T}^2 \times \mathbb{R}^2$. Here $\{f, g\} = \nabla_q f \cdot \nabla_p g - \nabla_q g \cdot \nabla_p f$ is the Poisson bracket of the functions f and g.

Now, we have the identities

$$\begin{aligned} \Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1} \circ \Psi_n^s &= \Lambda_n \circ \Gamma_{n+1} \circ \Psi_{n+1}^{\alpha_{n+1}^{-1} s} \circ \mathcal{T}_n^{-1} \\ &= \Lambda_n \circ \Phi_{n+1}^{t_{n+1}(\alpha_{n+1}^{-1} s)} \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1} = \Phi_n^{\theta'_n t_{n+1}(\alpha_{n+1}^{-1} s)} \circ \Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1}. \end{aligned} \tag{3.7.5}$$

Thus, formally, if Γ_{n+1} is an invariant torus with frequency vector ω_{n+1} of H_{n+1}, then

$$\Gamma_n = \Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1} =: \mathcal{M}_n(\Gamma_{n+1}) \tag{3.7.6}$$

is the invariant torus with frequency vector ω_n of H_n. The flow parameters t_n and t_{n+1} are related by $t_n(s) = \theta'_n t_{n+1}(\alpha_{n+1}^{-1} s)$. One solution of this functional equation is $t_n(s) = \xi s / \prod_{i=1}^{n} \tau_i$, where ξ is a constant.

3.7.2 Construction of invariant tori

The formal relationship (3.7.6) between invariant tori of a Hamiltonian and its renormalization image motivates the construction of an invariant torus. In this section we study the properties of the maps $\mathcal{M}_n$, defined by (3.7.6), and use them to construct invariant tori for near-integrable Hamiltonians in a space of functions of low regularity. We assume that $\omega_0 \in \mathcal{D}(\beta)$, with $0 \leq \beta < (\sqrt{161} - 11)/10$, and that $H_n \in \mathbb{I}_n^+ \mathcal{A}_n(\rho')$, $n \in \mathbb{N}_0$, is the renormalization orbit of a given Hamiltonian $H_0 \in B_{0,\rho'}^+(C^2)$ in the domain of attraction of the integrable limit set.

Introduce the coordinates $x_1 = \omega'_n \cdot q$, $x_2 = \Omega'_n \cdot q$, $y_1 = \hat{\omega}_n \cdot p$ and $y_2 = \Omega_n \cdot p$, for a given $n \in \mathbb{N}_0$. Denote the partial derivative with respect to x_1, x_2, y_1 and y_2 as ∂_i, for $i = 1, 2, 3$ and 4, respectively. We can define the components of a vector valued function $f = (f^q, f^p)$ on $D_{n,0}(\delta_1)$, $\delta_1 > 0$, as $f^1 = \omega'_n \cdot f^q$, $f^2 = \Omega'_n \cdot f^q$, $f^3 = \hat{\omega}_n \cdot f^p$, $f^4 = \Omega_n \cdot f^p$.

Definition 3.7.3 *Given $r > 0$ and $\delta_1 > 0$, let $\mathcal{B}_{n,0}(\delta_1)$ be the Banach space of vector-valued functions $f = (f^q, f^p)$ on $D_{n,0}(\delta_1)$, whose components f^i, $i = 1, 2, 3, 4$, belong to $\mathcal{A}_{n,0}(\delta_1)$, with the norm*

$$\|f\|_{n,\delta_1} = \max\{c_n \|f^1\|_{n,\delta_1}, \|f^2\|_{n,\delta_1}, \|f^3\|_{n,\delta_1}, \|f^4\|_{n,\delta_1}\}, \tag{3.7.7}$$

where $c_n = \tilde{A}_n \prod_{i=0}^{n-1} \|T_i^{-1} \Omega_i\|$.

The maps $\mathcal{M}_n$ are defined formally by the action on a vector valued function

$F = \mathrm{I} + f$, where I is the identity map and $f \in \mathcal{B}_{n+1,0}(\delta_1)$,

$$\mathcal{M}_n(F) = \Lambda_n \circ F \circ \mathcal{T}_n^{-1}. \tag{3.7.8}$$

The weight factor c_n has been introduced in the norm (3.7.7) in order to make the map $\mathcal{M}_n$ a contraction; if the norm were defined without that factor, the scaling map in Λ_n would actually expand the first component of F.

Similarly, for a vector valued function $g = (g^q, g^p)$ on $D_n(\rho)$ with components $g^1 = \omega_n' \cdot g^q$, $g^2 = \Omega_n' \cdot g^q$, $g^3 = \hat{\omega}_n \cdot g^p$ and $g^4 = \Omega_n \cdot g^p$ in $\mathcal{A}_n(\rho)$, define the norms

$$\|g\|_{n,\rho} = \max\{c_n\|g^1\|_{n,\rho}, \|g^2\|_{n,\rho}, \|g^3\|_{n,\rho}, \|g^4\|_{n,\rho}\}, \tag{3.7.9}$$

and

$$\begin{aligned}\|g\|'_{n,\rho} = \max\{&\|\partial_1 g^1\|_{n,\rho} + c_n \sum_{i=2}^{4} \|\partial_i g^1\|_{n,\rho}, c_n^{-1}\|\partial_1 g^2\|_{n,\rho} + \sum_{i=2}^{4} \|\partial_i g^2\|_{n,\rho},\\ &c_n^{-1}\|\partial_1 g^3\|_{n,\rho} + \sum_{i=2}^{4} \|\partial_i g^3\|_{n,\rho}, c_n^{-1}\|\partial_1 g^4\|_{n,\rho} + \sum_{i=2}^{4} \|\partial_i g^4\|_{n,\rho}\}.\end{aligned} \tag{3.7.10}$$

Before proving that in the above norm the maps $\mathcal{M}_n$ are contractions, let us state the following proposition.

Proposition 3.7.4 *Let $\delta_1 > 0$, $\delta_1^< = \delta_1\|\omega_{n+1}\|/(\alpha_{n+1}\|\omega_n\|) < 2\delta_1/3$ and $\rho > 0$, componentwise, with $\rho_1 > \delta_1$. Let also $f \in \mathcal{A}_{n+1,0}(\delta_1)$, $g, g' \in \mathcal{A}_{n,0}(\delta_1)$, $h \in \mathcal{A}_n(\rho)$ and $X, Y \in \mathcal{A}^2_{n,0}(\delta_1)$, such that $\delta_1 + \|\omega_n' \cdot X_n\|_{n,\delta_1} < \rho_1$, $\|\Omega_n' \cdot X\|_{n,\delta_1} < \rho_1$, $\|\hat{\omega}_n \cdot Y\|_{n,\delta_1} < \rho_2$ and $\|\Omega_n \cdot Y\|_{n,\delta_1} < \rho_2$. If $U(q,0) = (q + X(q,0), Y(q,0))$ is a given change of variables, then*

(i) $|g(q,0)| \le \|g\|_{n,\delta_1}$, $\forall (q,0) \in D_{n,0}(\delta_1)$,

(ii) $gg' \in \mathcal{A}_{n,0}(\delta_1)$ *and* $\|gg'\|_{n,\delta_1} \leq \|g\|_{n,\delta_1}\|g'\|_{n,\delta_1}$,

(iii) $\|f \circ \mathcal{T}_n^{-1}\|_{n,\delta_1} \leq \|T_n^{-1}\Omega_n\|^r \|f\|_{n+1,\delta_1^<}$,

(iv) $\|h \circ U\|_{n,\delta_1} \leq c_{\rho_1}(\|\Omega_n' \cdot X\|_{n,\delta_1})\|h\|_{n,\rho}$,

where $c_{\rho_1}(s) = \sup_{t \geq 0}(1+t)^r e^{-(\rho_1 - s)t}$, *for* $|s| < \rho_1$.

The proof of this proposition is straightforward and will be omitted. Denote by $B_n(b)$ an open ball of radius $b > 0$ in the affine space $\mathrm{I} + \mathcal{B}_{n,0}(\delta_1)$, centered at the identity.

Lemma 3.7.5 *Let* $r < 1$ *and let* $0 < \delta_1 < \rho_1'$. *For sufficiently small* $b, C > 0$ *and all Hamiltonians* H_n, $n \in \mathbb{N}_0$, *of the renormalization orbit of* $H_0 \in B^+_{0,\rho'}(C^2) \subset \mathbb{I}_0^+ \mathcal{A}_0(\rho')$, *the map* $\mathcal{M}_n$ *is a contraction with the contraction rate* $a < 1$ *(independent of* H_0 *and* n*) from* $B_{n+1}(b)$ *into* $B_n(b)$.

Proof: Let $F \in B_{n+1}(b)$ and let $f = F - \mathrm{I}$. Define the map $\mathcal{N}_n$ by

$$\mathcal{N}_n(f) = \mathcal{M}_n(F) - \mathrm{I} = \Lambda_n \circ (\mathrm{I} + f) \circ \mathcal{T}_n^{-1} - \mathrm{I}\,.$$

The fact that the maps $V_{H_n'}$, $U_{H_n''}$ and $S_{\tilde{H}_n}$, included in Λ_n, are close to the identity motivates us to write $\Lambda_n = \mathcal{T}_n \circ (\mathrm{I} + g_{n+1})$. Therefore,

$$\mathcal{N}_n(f) = \mathcal{T}_n \circ (f + g_{n+1} \circ (\mathrm{I} + f)) \circ \mathcal{T}_n^{-1}.$$

As the maps $V_{H_n'}$, $U_{H_n''}$ and $S_{\tilde{H}_n}$ depend only on the q-dependent part of the Hamiltonian, $\|g^i_{n+1}\|_{n+1,\rho'} \leq C_1\|(\mathbb{I} - \mathbb{E})h_n\|_{n,\rho'}$, for $i = 1, 2, 3, 4$, i.e. $\|g^i_{n+1}\|_{n+1,\rho'} \leq CC_1\tilde{A}_n^{-\tau}$, where $C_1 > 0$ is an n-independent constant. Here, we assume that $C > 0$ is sufficiently small such that the estimates of Theorem 3.6.4 are valid. Using the

Diophantine bound (3.6.9) and the inequalities (3.6.10), we find that, if $\tau \geq 2+\beta$, then

$$\|g_{n+1}\|_{n+1,\rho'} \leq \tilde{A}_{n+1} \prod_{i=0}^{n} \|T_i^{-1}\Omega_i\| \frac{CC_1}{\tilde{A}_n^{\tau}} \leq \frac{CC_2}{\tilde{A}_n^{\tau-2-\beta}} \leq CC_2,$$

where $C_2 > 0$ is a constant. These inequalities show that the growth of the weight factor with n is slower than the decrease of the norm of the q-dependent part of Hamiltonian.

The derivative of the operator $\mathcal{N}_n$ at f is given by its action on an arbitrary $\tilde{f} \in B_{n+1}(b)$,

$$D\mathcal{N}_n(f)\tilde{f} = \mathcal{T}_n \circ (\tilde{f} + Dg_{n+1} \circ (\mathrm{I}+f)\tilde{f}) \circ \mathcal{T}_n^{-1}.$$

Using the Proposition 3.7.4, we find that for $(\delta_1, 0) < \rho'' < \rho'$, componentwise,

$$\begin{aligned}\|D\mathcal{N}_n(f)\tilde{f}\|_{n,\delta_1} \leq \|T_n^{-1}\Omega_n\|^{r-1}\|\tilde{f}\|_{n+1,\delta_1^<} \\ + \|T_n^{-1}\Omega_n\|^{r-1} c_{\rho_1''}(\|f^2\|_{n+1,\delta_1^<})\|g_{n+1}\|'_{n+1,\rho''}\|\tilde{f}\|_{n+1,\delta_1^<},\end{aligned}$$

providing that $\delta_1^< + \|f^1\|_{n+1,\delta_1^<} < \rho_1''$, $\|f^2\|_{n+1,\delta_1^<} < \rho_1''$, $\|f^3\|_{n+1,\delta_1^<} < \rho_2''$ and $\|f^4\|_{n+1,\delta_1^<} < \rho_2''$. These conditions are satisfied for sufficiently small $b > 0$. Using Cauchy estimates on the norms of the derivatives $\|\partial_i g_{n+1}^j\|_{n+1,\rho''}$, for $i, j = 1, 2, 3, 4$, we find that $\|g_{n+1}\|'_{n+1,\rho''} \leq C_3\|g_{n+1}\|_{n+1,\rho'}$, with $C_3 > 0$. Thus, as $r < 1$ and $\|T_n^{-1}\Omega_n\| \geq 3/2$, for sufficiently small $C > 0$, the operator norm $\|D\mathcal{N}_n(f)\|$ can be bounded by a positive constant $a < 1$.

As $\|\mathcal{N}_n(0)\|_{n,\delta_1} \leq \|T_n^{-1}\Omega_n\|^{r-1} c_{\rho_1'}(0)\|g_{n+1}\|_{n+1,\rho'}$, for sufficiently small $C > 0$, $\|\mathcal{N}_n(0)\|_{n,\delta_1} < (1-a)b$. This shows that $\mathcal{M}_n$ maps $B_{n+1}(b)$ into $B_n(b)$ and contracts distances at least by a factor $a < 1$. **QED**

Before we prove that every Hamiltonian H_n, of the renormalization orbit of $H_0 \in B^+_{0,\rho'}(C^2)$, with sufficiently small $C > 0$, has an invariant torus of Diophantine frequency vector ω_n, we will prove the following proposition.

Proposition 3.7.6 *Let $0 < \delta_1 < \rho'_1$ and $\rho'_2 > 0$. For sufficiently small $b > 0$, the following holds. If $H_0 \in B^+_{0,\rho'}(C^2)$, $|t_m - s_m| \le C_4 \|(\mathbb{I} - \mathbb{E})H_m\|_{m,\rho'}$ and $|t_m| < C_5$, for some $C_4, C_5 > 0$, $t_m, s_m \in \mathbb{R}$ and for all $m \in \mathbb{N}_0$, and if $C > 0$ is sufficiently small, then*

$$\|(\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m} - \mathrm{I})\|_{m,\delta_1} \le b, \tag{3.7.11}$$

where $\Phi_{m,0}$ is the flow for the Hamiltonian H_m^0 restricted to $D_{m,0}(\delta_1)$.

Proof: Let $s \in \mathbb{R}$ be given and let $(q,0) \in D_{m,0}(\delta_1)$. The flow Φ_m for the Hamiltonian H_m satisfies the equation,

$$\frac{d}{dt}(\Phi_m^t \circ \Phi_{m,0}^{-s} - \mathrm{I})(q,0) = (\mathbb{J} \cdot \nabla H_m \circ \Phi_m^t \circ \Phi_{m,0}^{-s})(q,0), \tag{3.7.12}$$

where $\mathbb{J}(q,p) = (p,-q)$ and $\nabla = (\nabla_q, \nabla_p)$. Introducing the function

$$\Upsilon_s(q,t) = (\Phi_m^t \circ \Phi_{m,0}^{-s} - \mathrm{I})(q,0),$$

2π-periodic in both q-variables, we can integrate the equation (3.7.12) to obtain the integral equation

$$\Upsilon_s(q,t) = -(s\omega_m, 0) + \int_0^t dt' \{\mathbb{J} \cdot \nabla H_m \circ [(q,0) + \Upsilon_s(q,t')]\}. \tag{3.7.13}$$

This equation can be viewed as the fixed point equation of the functional Ξ_s, defined by the action

$$\Xi_s(\Upsilon_s(q,t), q) = -(s\omega_m, 0) + \int_0^t dt' \{\mathbb{J} \cdot \nabla H_m \circ [(q,0) + \Upsilon_s(q,t')]\}, \tag{3.7.14}$$

on a space of functions $\Upsilon_s : D_{m,0}(\delta_1) \times J \to \mathbb{C}^2 \times \mathbb{C}^2$, $J = [0, t] \subset \mathbb{R}$, with the norm

$$|||\Upsilon_s||| = \sup_{t' \in J} \max\{c_m \|\Upsilon_s^1(\cdot, t')\|_{m,\delta_1}, \|\Upsilon_s^2(\cdot, t')\|_{m,\delta_1}, c_m \|\Upsilon_s^3(\cdot, t')\|_{m,\delta_1}, c_m \|\Upsilon_s^4(\cdot, t')\|_{m,\delta_1}\},$$

where $\Upsilon_s^1 = \omega_m' \cdot \Upsilon_s^q, \Upsilon_s^2 = \Omega_m' \cdot \Upsilon_s^q$, $\Upsilon_s^3 = \hat{\omega}_m \cdot \Upsilon_s^p, \Upsilon_s^4 = \Omega_m \cdot \Upsilon_s^p$, and $\Upsilon_s = (\Upsilon_s^q, \Upsilon_s^p)$.

We consider an open ball of functions Υ_s in that space that satisfy $|||\Upsilon_s||| < b$, with $0 < b < \min\{\rho_1' - \delta_1, \rho_2\}$. This justifies the formal use of equations (3.7.13) and (3.7.14), as for any given $t' \in J$, one has $\|\Upsilon_s(\cdot, t')\|_{m,\delta_1} \leq |||\Upsilon_s|||$.

Now, if $\Upsilon_s^{(1)}$ and $\Upsilon_s^{(2)}$ are two functions from that ball, we have

$$\Xi_s(\Upsilon_s^{(2)}(q, t), q) - \Xi_s(\Upsilon_s^{(1)}(q, t), q) = \int_0^t dt' \int_{\Upsilon_s^{(1)}(q,t')}^{\Upsilon_s^{(2)}(q,t')} d\Upsilon \cdot \nabla(\mathbb{J} \cdot \nabla H_m) \circ [(q, 0) + \Upsilon].$$

Thus,

$$\left|\left|\left|\Xi_s(\Upsilon_s^{(2)}, \cdot) - \Xi_s(\Upsilon_s^{(1)}, \cdot)\right|\right|\right| \leq |t| \cdot \|\Upsilon_s^{(2)} - \Upsilon_s^{(1)}\| \cdot \|\nabla(\mathbb{J} \cdot \nabla H_m) \circ [(q, 0) + \Upsilon]\|, \quad (3.7.15)$$

where

$$\|\nabla(\mathbb{J} \cdot \nabla H_m) \circ [\Upsilon + (q, 0)]\| \leq \max\{c_m \|\partial_2 \partial_3 H_m\|_{m,\rho''} + \sum_{i=1, i\neq 2}^{4} \|\partial_i \partial_3 H_m\|_{m,\rho''},$$
$$\|\partial_2 \partial_4 H_m\|_{m,\rho''} + c_m^{-1} \sum_{i=1, i\neq 2}^{4} \|\partial_i \partial_4 H_m\|_{m,\rho''}, c_m \|\partial_2 \partial_1 H_m\|_{m,\rho''} + \sum_{i=1, i\neq 2}^{4} \|\partial_i \partial_1 H_m\|_{m,\rho''},$$
$$c_m \|\partial_2 \partial_2 H_m\|_{m,\rho''} + \sum_{i=1, i\neq 2}^{4} \|\partial_i \partial_2 H_m\|_{m,\rho''}\},$$

with $b + \delta_1 < \rho_1'' < \rho_1'$ and $b < \rho_2'' < \rho_2'$.

Using Cauchy estimates on the norms of the derivatives and the fact that $\|\partial_2 \partial_i H_m\|_{m,\rho''} \leq C_6 \|(\mathbb{I} - \mathbb{E}) H_m\|_{m,\rho'}$, for $i = 1, 2, 3$, where $C_6 > 0$, this norm can be bounded by an m-independent constant. Here, we again use the fact that

$\|(\mathbb{I}-\mathbb{E})H_m\|_{m,\rho'}$ drops faster with m, than c_m increases. The inequality (3.7.15) then implies that, if $|t| = |t_m| < C_5$, $H_0 \in B^+_{0,\rho'}(C^2)$ and $C > 0$ is sufficiently small, Ξ_s is a contraction with a contraction rate $\bar{a} < 1$.

Notice that

$$\Xi_s(0,q) = ((t-s)\omega_m, 0) + \mathbb{J} \cdot \nabla h_m(q,0)t,$$

and that $\nabla \mathbb{E} h_m(q,0) = 0$. Therefore, since $|t_m - s_m| \leq C_4 \|(\mathbb{I}-\mathbb{E})H_m\|_{m,\rho'}$, where $C_4 > 0$ and $|t_m| < C_5$, if $t = t_m$ and $s = s_m$, then, for sufficiently small $C > 0$, we have $|||\Xi_{s_m}(0,q)||| < (1-\bar{a})b$. This shows that $\Xi_{s_m}(\cdot, q)$ is a contraction on a ball of radius $b > 0$, and has a unique fixed point Υ_{s_m}, of norm $|||\Upsilon_{s_m}||| < b$, that satisfies the equation (3.7.13). The claim follows from the fact that for $|t_m| < C_5$, one has $\|\Upsilon_{s_m}(\cdot, t_m)\|_{m,\delta_1} \leq |||\Upsilon_{s_m}|||$. **QED**

Let, in the following, $b > 0$ and $C > 0$ be chosen sufficiently small such that the assumptions of Lemma 3.7.5 and Proposition 3.7.6 are satisfied. Let F_m, $m \in \mathbb{N}_0$, be arbitrary maps in $B_m(b)$. Define

$$\Gamma_{n,m} = (\mathcal{M}_n \circ \cdots \circ \mathcal{M}_{m-1})(F_m), \tag{3.7.16}$$

for $0 \leq n < m$, or, taking into account the Definition (3.7.8) of the operators $\mathcal{M}_n$,

$$\Gamma_{n,m} = \Lambda_n \circ \cdots \circ \Lambda_{m-1} \circ F_m \circ \mathcal{T}^{-1}_{m-1} \circ \cdots \circ \mathcal{T}^{-1}_n. \tag{3.7.17}$$

Here, the maps Λ_n and the operators $\mathcal{M}_n$ are associated to the sequence of Hamiltonians H_n, $n \in \mathbb{N}_0$.

Theorem 3.7.7 *Let $0 < \delta_1 < \rho'_1$. If $H_0 \in B^+_{0,\rho'}(C^2)$ and $b, C > 0$ are sufficiently* *small, the limits* $\Gamma_n = \lim_{m\to\infty} \Gamma_{n,m}$*,* $n \in \mathbb{N}_0$*, exist in* $B_n(b)$*, and satisfy*

$$\|\Gamma_n - \mathrm{I}\|_{n,\delta_1} \leq b. \tag{3.7.18}$$

Γ_n *is an invariant torus of* H_n *with frequency vector* ω_n*. The invariant tori satisfy*

$$\Gamma_n = \Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1}. \tag{3.7.19}$$

Proof: By Lemma 3.7.5, if $n < m < i$, then $\|\Gamma_{n,i} - \Gamma_{n,m}\|_{n,\delta_1} \leq 2ba^{m-n}$. Thus, the sequence $m \mapsto \Gamma_{n,m}$ is Cauchy and converges in this norm to a limit Γ_n. Moreover, this limit is independent of F_m. The relationship (3.7.19) follows from the continuity of $\mathcal{M}_n$.

Now, we want to show that, in the above norm, $\Gamma_n \to \mathrm{I}$, when $n \to \infty$. Consider a one-parameter family of Hamiltonians $H_n(s) = H_n^0 + s(H_n - H_n^0)$, $s \in \mathbb{C}$, and let $\Lambda_n(s)$ and $\Gamma_n(s)$ be the corresponding one parameter families of maps. The map $s \mapsto \Lambda_n(s)$ is analytic in the domain $|s| < \zeta_n \tilde{A}_{n-1}^2/C$ containing the unit disc. By uniform convergence, $s \mapsto \Gamma_n(s)$ is analytic from the same domain into $B_n(b)$. As $\Gamma_n(0) = \mathrm{I}$, by Schwartz's lemma, $\|\Gamma_n(1) - \mathrm{I}\|_{n,\delta_1} \leq 2bC/(\zeta_n \tilde{A}_{n-1}^2)$. The inequality (3.7.18) then follows from Theorem 3.6.4.

It remains to be proved that Γ_n is an invariant torus of H_n with frequency vector ω_n. First, we formally have the identities

$$\begin{aligned}
\Phi_n^{t_n} \circ \Gamma_{n,m} \circ \Psi_n^{-s_n} &= \Phi_n^{t_n} \circ \Lambda_n \circ \cdots \circ \Lambda_{m-1} \circ \mathcal{T}_{m-1}^{-1} \circ \cdots \circ \mathcal{T}_n^{-1} \circ \Psi_n^{-s_n} \\
&= \Lambda_n \circ \Phi_{n+1}^{\theta_n'^{-1} t_n} \circ \cdots \circ \Lambda_{m-1} \circ \mathcal{T}_{m-1}^{-1} \circ \cdots \circ \Psi_{n+1}^{-\alpha_{n+1}^{-1} s_n} \circ \mathcal{T}_n^{-1} \\
&= \Lambda_n \circ \cdots \circ \Lambda_{m-1} \circ \Phi_m^{t_m} \circ \Psi_m^{-s_m} \circ \mathcal{T}_{m-1}^{-1} \circ \cdots \circ \mathcal{T}_n^{-1},
\end{aligned} \tag{3.7.20}$$

on any domain where the compositions are well-defined, where $t_m = \theta'^{-1}_{m-1} \cdots \theta'^{-1}_0 t_0$ and $s_m = \alpha_m^{-1} \cdots \alpha_1^{-1} s_0$, $m \in \mathbb{N}$. As the map Λ_n maps $D_n(\rho')$ into $D_n(\rho'')$, with $\rho'' < \rho'$, componentwise, for t_0 in an open interval containing zero and H_0 sufficiently close to H_0^0, the equality $\Phi_n^{t_n} \circ \Lambda_n = \Lambda_n \circ \Phi_{n+1}^{\theta_n'^{-1} t_n}$, $n \in \mathbb{N}_0$, is an identity between maps on $D_n(\rho')$, with a range contained in $D_n(\rho')$.

As both sides of (3.7.20) are the maps on $\mathcal{B}_{n,0}(\delta_1)$, we have the identity

$$\Phi_n^{t_n} \circ \Gamma_{n,m} \circ \Psi_n^{-s_n} = (\mathcal{M}_n \circ \cdots \circ \mathcal{M}_{m-1})(\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m}).$$

Here, we have also used the fact that I is the invariant torus of frequency ω_m of H_m^0. It suffices to show that the map $\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m}$ belongs to the domain of $\mathcal{M}_{m-1}$, for sufficiently large m.

Let $t_0 = \xi s_0$, where $\xi = \prod_{j=1}^{\infty} \tau_j = \lim_{k\to\infty} \prod_{j=1}^{k} \tau_j$ and $\tau_j = \tau_{\tilde{H}_j}$. The existence of this limit follows from the convergence of $\sum_{j=0}^{\infty} |\tau_{j+1} - 1|$ and the mean value theorem. The convergence of the sum itself follows from the fact that $|\tau_{j+1} - 1| \leq C_7 \|(\mathbb{I} - \mathbb{E}) h_j\|_{j,\rho'}$, where $C_7 > 0$, and Theorem 3.6.4. We have,

$$\frac{\tilde{A}_m}{\alpha_0}(t_m - s_m) = t_0 \prod_{i=1}^{m} \tau_i^{-1} - s_0 = \left(\prod_{i=m+1}^{\infty} \tau_i - 1 \right) s_0 \leq s_0 C_8 \|(\mathbb{I} - \mathbb{E}) H_m\|_{m,\rho'},$$

with $C_8 > 0$.

Thus, for sufficiently small $b, C > 0$, the assumptions of Proposition 3.7.6 are satisfied. The same proposition then implies that $\|\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m} - \mathrm{I}\|_{m,\delta_1} < b$ and that the map $\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m}$ belongs to the domain of $\mathcal{M}_{m-1}$. The right hand side of (3.7.20) converges in $\mathcal{B}_{n,0}(\delta_1)$ to Γ_n when $m \to \infty$. As the convergence implies pointwise convergence and since the maps $\Phi_n^{t_n}$ and $\Psi_n^{s_n}$ are both continuous and invertible, we conclude that $\Phi_n^{t_n} \circ \Gamma_n \circ \Psi_n^{-s_n} = \Gamma_n$. Thus, Γ_n is an invariant torus of frequency ω_n, associated to the Hamiltonian H_n. **QED**

3.7.3 Analyticity of invariant tori

Theorem 3.7.7 shows that every Hamiltonian H_0 sufficiently close to H_0^0 has an invariant torus Γ_0 with a Diophantine frequency vector ω_0. In this section, we show that the so-constructed invariant tori can be extended to analytic functions. We use the fact that Γ_0 depends analytically on H_0.

Define the translation $I_v : (q,p) \mapsto (q+v,p)$, for $v \in \mathbb{R}^2$, a one parameter family of Hamiltonians $H_n(v) = H_n \circ I_v$ and the map $\mathcal{I}_v : H_n \mapsto H_n \circ I_v^{-1}$. Let $\Lambda_n(v)$ be the one parameter family of maps introduced in Definition 3.4.1, associated to $H_n(v)$. Let $0 < \rho' < \varrho' < \varrho < \rho < 3\rho'/2$, componentwise, be chosen as in Definition 3.4.1.

Proposition 3.7.8 *Let $v \in \mathbb{R}^2$. For sufficiently small $c > 0$ and for all Hamiltonians H in the ball $B_{n,\varrho}(c) \subset \mathcal{A}_n(\varrho)$, of radius c, centered at H_n^0, the identity $U_H \circ I_v = I_v \circ U_{H\circ I_v}$, is valid on $D_n(\varrho')$, $n \in \mathbb{N}$.*

Proof: The proof of this proposition follows from the construction of the transformation U_H that satisfies the equation $\mathbb{I}^- H \circ U_H = 0$. In Section 3.5, this transformation is constructed as the composition of a sequence of canonical transformations U_ϕ generated by functions ϕ. As the translation $\mathcal{I}_v$ commutes with $\mathbb{I}^\pm$ and differentiation, we have $\phi_{H\circ I_v} = \phi_H \circ I_v$, where ϕ_H is the generating function associated to H. Then, $U_{\phi_H} \circ I_v = I_v \circ U_{\phi_{H\circ I_v}}$. The claim follows, by taking the limit of the product of the above mentioned sequence. **QED**

Proposition 3.7.9 *Let $v \in \mathbb{R}^2$. For sufficiently small $C > 0$ and for all Hamiltonians $H_0 \in B^+_{0,\rho'}(C^2) \subset \mathbb{I}_0^+ \mathcal{A}_n(\rho')$, if the sequence H_n, $n \in \mathbb{N}_0$, is the renor-*

malization orbit of H_0*, then the identity* $\Lambda_n \circ I_v = I_{T_n v} \circ \Lambda_n(T_n v)$*, is valid on* $D_n(\rho')$*,* $n \in \mathbb{N}$*.*

Proof: Notice first that $\mathcal{T}_n \circ I_v = I_{T_n v} \circ \mathcal{T}_n$ and that $V_{H_n'}$ and $S_{\tilde{H}_n}$ commute with I_v. Further, notice that $V_{H_n' \circ I_v} = V_{H_n'}$ and $S_{\tilde{H}_n \circ I_v} = S_{\tilde{H}_n}$. Recall that $\Lambda_n = \mathcal{T}_n \circ V_{H_n'} \circ U_{H_n''} \circ S_{\tilde{H}_n}$, $H_n' = (\theta_n/\mu_n) H_n \circ \mathcal{T}_n$, $H_n'' = H_n' \circ V_{H_n'}$ and $\tilde{H}_n = H_n'' \circ U_{H_n''}$. Let $H_n'(v) = (\theta_n/\mu_n) H_n(v) \circ \mathcal{T}_n$, $H_n''(v) = H_n'(v) \circ V_{H_n'(v)}$ and $\tilde{H}_n(v) = H_n''(v) \circ U_{H_n''(v)}$. Using the previous relations and the Proposition 3.7.8, we find that $\Lambda_n \circ I_v = I_{T_n v} \circ \mathcal{T}_n \circ V_{H_n'(T_n v)} \circ U_{H_n''(T_n v)} \circ S_{\tilde{H}_n(T_n v)}$, and the claim immediately follows. **QED**

Proposition 3.7.10 *Let* $v \in \mathbb{R}^2$*. For sufficiently small* $C > 0$ *and for all Hamiltonians* $H_0 \in B^+_{0,\rho'}(C^2) \subset \mathbb{I}^+_0 \mathcal{A}_n(\rho')$*, we have* $I_v \circ \Gamma_0(H_0 \circ I_v) = \Gamma_0(H_0) \circ I_v$*. Here, by writing* $\Gamma_0(H_0)$*, we have emphasized that the invariant torus* Γ_0 *is associated to a Hamiltonian* H_0*.*

Proof: Let $v \in \mathbb{R}^2$ be fixed. Since $\mathcal{I}_v^{-1}$ is an isometry on $\mathcal{A}_n(\rho)$, for sufficiently small $C > 0$ and for all Hamiltonians $H_0 \in B^+_{0,\rho'}(C^2)$, $\mathcal{I}_v^{-1}(H_0)$ belongs to $B^+_{0,\rho'}(C^2)$ and the orbit $\mathcal{R}_{n-1} \circ \cdots \circ \mathcal{R}_0(\mathcal{I}_v^{-1}(H_0))$, $n \in \mathbb{N}$, satisfies the bounds analogous to those obtained in Theorem 3.6.4. Thus, we can construct the maps

$$\Gamma_{0,m}(H_0 \circ I_v) = \Lambda_0(v) \circ \cdots \circ \Lambda_{m-1}(T_{m-2}^{-1} \ldots T_0^{-1} v) \circ \mathcal{T}_{m-1}^{-1} \circ \cdots \circ \mathcal{T}_0^{-1},$$

for $m \in \mathbb{N}$. By Theorem 3.7.7, the limit $\Gamma_0(H_0 \circ I_v) = \lim_{m \to \infty} \Gamma_{0,m}(H_0 \circ I_v)$ is an invariant torus of $H_0 \circ I_v$ with frequency vector ω_0. Using Proposition 3.7.9, we find

$$\begin{aligned}
\Gamma_{0,m} \circ I_v &= \Lambda_0 \circ \cdots \circ \Lambda_{m-1} \circ \mathcal{T}_{m-1}^{-1} \circ \cdots \circ \mathcal{T}_0^{-1} \circ I_v \\
&= I_v \circ \Lambda_0(v) \circ \cdots \circ \Lambda_{m-1}(T_{m-2}^{-1} \ldots T_0^{-1} v) \circ \mathcal{T}_{m-1}^{-1} \circ \cdots \circ \mathcal{T}_0^{-1} \\
&= I_v \circ \Gamma_{0,m}(H_0 \circ I_v).
\end{aligned}$$

Further, by taking the limit $m \to \infty$ of this identity, we find that $I_v \circ \Gamma_0(H_0 \circ I_v) = \Gamma_0(H_0) \circ I_v$. This holds for all Hamiltonians H_0 in a ball $B^+_{0,\rho'}(C^2)$ independent of $v \in \mathbb{R}^2$. **QED**

The relationship between $\Gamma_0(H_0 \circ I_v)$ and $\Gamma_0(H_0)$, valid *a priori* for $v \in \mathbb{R}^2$, can be used to analytically continue $\Gamma_0(H_0)$ away from $\mathbb{R}^2 \times \{0\}$.

In the following, given any two sets X and Y, if $x \in X$, then $\mathcal{E}_x$ denotes the evaluation functional defined by $\mathcal{E}_x f = f(x)$, for all functions $f \in Y^X$.

Theorem 3.7.11 *Let $\rho_1 - \rho_1' > \delta_1 > \delta_1' > 0$ and $\rho_2 - \rho_2' > 0$. There exists an open neighborhood B of H_0^0 in $\mathcal{A}_0(\rho)$, such that for every Hamiltonian $H_0 \in B$, the map*

$$\mathcal{G}_{H_0}(q,p) = \mathcal{E}_0 I_q \circ \Gamma_0(H_0 \circ I_q), \qquad q \in D_{0,1}(\delta_1'), \quad p \in \mathbb{C}^2, \tag{3.7.21}$$

defines an analytic function $\mathcal{G}_{H_0}$ on $D_{0,1}(\delta_1') \times \mathbb{C}^2$ whose restriction to $\mathbb{R}^2 \times \{0\}$ coincides with $\Gamma_0(H_0)$.

Proof: Let $\delta = (\delta_1, 0)$. The map $(q, H) \mapsto H \circ I_q$ is analytic from $D_{0,1}(\delta_1) \times \mathcal{A}_0(\rho)$ into $\mathcal{A}_0(\rho')$. This follows from the fact that the map $\mathcal{I}_q^{-1}$ is bounded with norm 1 from $\mathcal{A}_0(\rho' + \delta)$ to $\mathcal{A}_0(\rho')$, for every $q \in D_{0,1}(\delta_1)$, and the differentiation is bounded from $\mathcal{A}_0(\rho)$ to $\mathcal{A}_0(\rho' + \delta)$.

Define the map $G : H_0 \mapsto G(H_0)$, by setting $G(H_0)(q,p) = \Gamma_0(H_0)(q,0) - S_0(q,p)$, for H_0 in the domain of Γ_0. Here $S_0(q,p) = (q,0)$. As $\mathcal{I}_q^{-1}$ is bounded from $\mathcal{A}_0(\rho)$ to $\mathcal{A}_0(\rho')$, there exists an open ball B in $\mathcal{A}_0(\rho)$ containing H_0^0, such that for every $q \in D_{0,1}(\delta_1)$, both H_0 and $\mathcal{I}_q^{-1} H_0$ belong to the domain of Γ_0. Since Γ_0 depends analytically on H_0, so does G, and as, $\mathcal{E}_0$ is bounded, the map $(q, H_0) \mapsto \mathcal{E}_0 G(\mathcal{I}_q^{-1} H_0)$ is analytic on $D_{0,1}(\delta_1) \times B$. The analyticity of $\mathcal{G}_{H_0}$ for $H_0 \in B$ follows from the identity $(\mathcal{G}_{H_0} - S_0)(q,p) = \mathcal{E}_0 G(\mathcal{I}_q^{-1} H_0)$.

Using Proposition 3.7.10, we find

$$\mathcal{E}_{(q,0)}\Gamma_0(H_0) = \mathcal{E}_0(\Gamma_0(H_0) \circ I_q) = \mathcal{E}_0 I_q \circ \Gamma_0(H_0 \circ I_q) = \mathcal{G}_{H_0}(q,0),$$

for all $q \in \mathbb{R}^2$ and $H_0 \in B$. This shows that $\Gamma_0(H_0)$ and $\mathcal{G}_{H_0}$ agree on $\mathbb{R}^2 \times \{0\}$ as claimed. **QED**

Chapter 4

Renormalization of vector fields

In this chapter, we construct a renormalization scheme for vector fields that applies to Diophantine frequency vectors. The scheme preserves the important geometrical classes of vector fields. Every Diophantine frequency vector ω determines an analytic manifold $\mathcal{W}$ of infinitely renormalizable vector fields and every vector field on $\mathcal{W}$ has an analytic elliptic invariant torus with frequency vector ω. In the context of existence of invariant tori in parametrized families of vector fields, we discuss the non-degeneracy conditions and the corresponding reduction of the number of parameters. The results of some sections, such as those concerning the normal form theorem and the stable manifold theorem, are sufficiently general to be applied when considering a larger set of frequency vectors.

4.1 Introduction and main results

The existence of invariant tori that constrain the dynamics of near-integrable systems is not only a feature of Hamiltonian systems. These tori also exist in the phase space of a generic near-integrable vector field. Some areas of science, such as the theory of magnetic confinement in plasma physics, are based on this fact [36, 49]. This motivates the idea of extending the renormalization scheme for Hamiltonians to general vector fields.

In this chapter we construct a renormalization scheme for analytic vector fields on $\mathbb{T}^d \times \mathbb{R}^\ell$ that can be applied to the problem of persistence of quasi-periodic motion with Diophantine frequency vectors. The scheme applies to general vector fields and leaves invariant certain geometrical types, that we refer to as classes.

The considered classes of vector fields are:

(i) volume-preserving vector fields,

(ii) Hamiltonian vector fields,

(iii) vector fields symmetric with respect to $G : (x, y) \mapsto (x, -y)$,

(iv) vector fields reversible with respect to the involution $G : (x, y) \mapsto (-x, y)$.

With the exception of reversible vector fields, the other mentioned classes form Lie-subalgebras of the algebra of vector fields with respect to the commutator, defined for any pair of vector fields X and Y by $[X, Y] = (DY)X - (DX)Y$.

As an application of this renormalization scheme, we obtain KAM type results for each of the above classes of vector fields. We prove the existence of invariant tori with Diophantine frequency vectors in near-integrable families of vector fields. Similar results have been obtained by Moser [67] by different (KAM) methods.

Our analysis applies to vector fields that are close (after a change of variables, if necessary) to the vector field $K = (\omega, 0)$, with the constant vector $\omega \in \mathbb{R}^d$. This vector field is integrable, meaning its dynamics is constrained to invariant tori, and the flow on each invariant torus is characterized by the frequency vector $\omega \in \mathbb{R}^d$. We assume that ω satisfies a Diophantine condition, i.e. that there exist constants

$\mathcal{C} > 0$ and $\beta \geq 0$, such that

$$|\omega \cdot \nu| \geq \mathcal{C}\|\nu\|^{-(d-1+\beta)} \qquad (\forall\, \nu \in \mathbb{Z}^d/\{0\}). \tag{4.1.1}$$

The vector fields are assumed to be analytic on a complex neighborhood D_ρ of $D_0 = \mathbb{T}^d \times \{0\}$, characterized by the conditions $|\operatorname{Im} x_i| < \rho$ and $|y_j| < \rho$, where $(x, y) \in \mathbb{C}^d \times \mathbb{C}^\ell$. For any given $\rho > 0$, Banach spaces of such vector fields will be denoted by $\mathcal{A}_\rho$.

We start by describing briefly the renormalization scheme. In this chapter, we follow the general idea from the previous chapter, which is to take a continued fractions algorithm, acting on a set of frequency vectors, and to "lift" it to a space of functions that we renormalize, in this case vector fields, in some appropriate way. Since the dimension of the angular part of the phase space $d \geq 2$ is not restricted from above, we need a multidimensional generalization of the one-dimensional continued fraction algorithm. We choose a multidimensional continued fraction expansion [41] which, starting from a Diophantine vector $\omega_0 \in \mathbb{R}^d$, produces a sequence of vectors $\omega_n = \eta_n^{-1} T_n^{-1} \omega_{n-1}$, with $n \in \mathbb{N}$, where T_n is a matrix in $\mathrm{SL}(d, \mathbb{Z})$ and η_n is an appropriate normalization constant. The matrices T_n can be used to construct successive rational approximants to ω_0. Our n-th step renormalization group transformation $\mathcal{R}_n$, that corresponds to the matrix T_n^{-1}, has the property that it maps $K_{n-1} = (\omega_{n-1}, 0)$ into $K_n = (\omega_n, 0)$ (Figure 4.1). The precise definition is given below.

A one-step renormalization transformation involves a "scaling" of the torus variable x by a matrix $T \in \mathrm{SL}(d, \mathbb{Z})$, whose transpose is strongly contracting on the orthogonal complement of some unit vector $\omega \in \mathbb{R}^d$. Given such a matrix T, and a nonzero real number μ, define

$$\mathcal{T}(x,y) = (Tx, \bar{T}y)\,, \qquad \mathcal{S}_\mu(x,y) = (x, \mu y)\,, \tag{4.1.2}$$

where $(x,y) \in \mathbb{T}^d \times \mathbb{R}^\ell$. Here, $\bar{T}$ is either the $\ell \times \ell$ identity matrix, or if $\ell = d$ it is the inverse of the transpose of T. By choosing $\bar{T} = (T^\star)^{-1}$, in the case $\ell = d$, the Hamiltonian structure of the Hamiltonian vector fields is preserved with respect to the same symplectic form. The scaling of a vector field X on $\mathcal{M}$ is then given by $\mathcal{T}_\mu^* X$, the pullback of X under $\mathcal{T}_\mu = \mathcal{S}_\mu \circ \mathcal{T}$. Recall that the pullback of a vector field X, defined on the range of a differentiable map U, is given by $U^*X = (DU)^{-1}(X \circ U)$.

Notice that scaling by $\mathcal{T}_\mu^*$ is a singular operation on spaces of analytic vector fields, since it shrinks the domain of analyticity in the expanding direction of T. Although the domain loss is not small, it is possible to associate with $X \in \mathcal{A}_\varrho$ a change of variables $\mathcal{U}_X$, which is close to the identity for X close to $K = (\omega, 0)$, such that the renormalized vector field

$$\mathcal{R}(X) = \eta^{-1}\mathcal{T}^*\mathcal{S}_\mu^*\mathcal{U}_X^* X\,, \tag{4.1.3}$$

belongs again to $\mathcal{A}_\varrho$.

To be more specific, we will identify in Section 4.3 a subspace of "resonant" vector fields, containing K, such that the restriction of $\mathcal{T}_\mu^*$ to that subspace is compact, and in fact analyticity improving, for small $\mu > 0$. Then, using a general result from Section 4.4, we show that there exists an analytic map $X \mapsto \mathcal{U}_X$, defined near K, which makes $\mathcal{U}_X^* X$ resonant. In other words, the resonant vector fields, which behave well under scaling, can be regarded as a local normal form for vector fields. We note that $\mathcal{U}_K$ is the identity, so the transformation $\mathcal{R}$ maps K to $\widetilde{K} = (\widetilde{\omega}, 0)$, where $\widetilde{\omega} = \eta^{-1}T^{-1}\omega$.

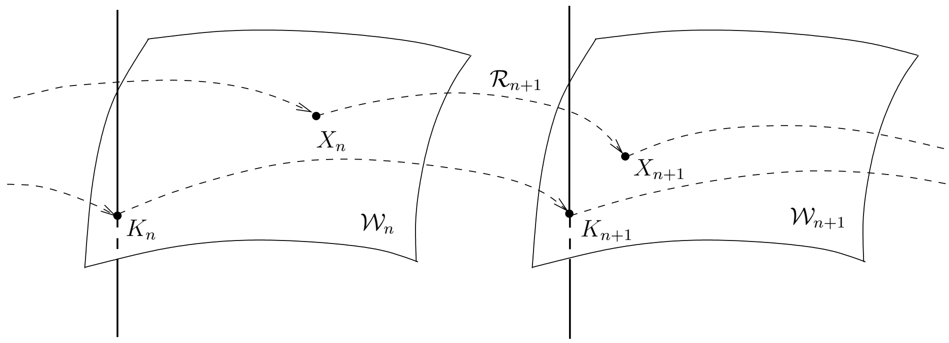

Figure 4.1: *Renormalization of vector fields. Subscript n on X, K and $\mathcal{W}$ stands for their n-th renormalization image.*

Theorem 4.1.1 *Let $\varrho > 0$. Given a Diophantine unit vector $\omega_0 \in \mathbb{R}^d$, there exists a sequence of matrices $T_n \in \mathrm{SL}(d, \mathbb{Z})$, and a corresponding sequence of transformations $\mathcal{R}_n$ of the form (4.1.3), such that the following holds. Define $\omega_n = \eta_n^{-1} T_n \omega_{n-1}$, for $n \in \mathbb{N}$, with $\eta_n > 0$ chosen in such a way that ω_n is a unit vector. Then $\mathcal{R}_n$ is well-defined and analytic in some open neighborhood $\mathcal{D}_{n-1}$ of $K_{n-1} = (\omega_{n-1}, 0)$ in $\mathcal{A}_\varrho$. The set $\mathcal{W}$ of infinitely renormalizable vector fields X_0 in $\mathcal{D}_0$, characterized by the property that $X_n = \mathcal{R}_n(X_{n-1})$ belongs to $\mathcal{D}_n$ for $n \in \mathbb{N}$, is the graph of an analytic function $W : (\mathbb{I} - \mathbb{P})\mathcal{D}_0 \to \mathbb{P}\mathcal{D}_0$, satisfying $W(0) = K$ and $DW(0) = 0$. The restriction of W to a particular class of vector fields takes values in that class.*

The proof of this theorem follows from Theorem 4.6.2, and the discussion following its proof, regarding the restriction of W to a particular class.

The set $\mathcal{W}$ can be regarded as the (local) stable manifold for the sequence of transformations $\{\mathcal{R}_n\}$, $n \in \mathbb{N}$. A stable manifold theorem that applies to such sequences of maps will be proved in Section 4.7.

Denote by A^u the space of all vector fields of the form $Y(x, y) = (u, My + v)$, with (u, v) a vector in $\mathbb{C}^d \times \mathbb{C}^\ell$ and M a complex $\ell \times \ell$ matrix. In Section 4.2, we will introduce the projection operator $\mathbb{P}$ from $\mathcal{A}_\rho$ onto the subspace A^u. The subspace of functions in $\mathcal{A}_\rho$ that do not depend on the coordinate $y \in \mathbb{C}^\ell$ will be denoted by $\mathcal{A}_\rho^0$. A function will be called "real" if it takes real values for real arguments.

Theorem 4.1.2 *Let $K = (\omega, 0)$, with $\omega \in \mathbb{R}^d$ Diophantine. Given $\rho > \delta > 0$, there exists an open neighborhood B of K in $\mathcal{A}_\rho$, and a real analytic map W with the properties described in Theorem 4.1.1 (assuming $\varrho + \delta < \rho$, $\omega_0 = \omega$ and $\mathcal{D}_0 = B$), such that the following holds. If $\mathcal{W}$ is the graph of W, then every vector field $X \in \mathcal{W}$ has an elliptic invariant torus $\Gamma_X \in \mathcal{A}_\delta^0$ with frequency vector ω. The map $X \mapsto \Gamma_X$ is real analytic on $\mathcal{W}$.*

This theorem is a consequence of Theorem 4.1.1, and Theorem 4.8.4 and Theorem 4.8.5 concerning the existence and analyticity of invariant tori for vector fields on $\mathcal{W}$. We note that the tori constructed here are elliptic in the sense that they have zero Lyapunov exponents. It should be possible to adapt this method to hyperbolic situations, but we will not pursue this question here.

Another remark is that the size of the neighborhood B is independent of ω, given the Diophantine constants and a lower bound on the norm of ω. We note that, for any fixed $\beta > 0$, the measure of the set of vectors ω that violate the Diophantine condition 4.1.1 approaches zero as the Diophantine constant $\mathcal{C}$ approaches zero [11].

In what follows, $\mathcal{H}_\rho$ denotes either $\mathcal{A}_\rho$, or the subspace of $\mathcal{A}_\rho$ consisting of all vector fields in a given class. The intersection of A^u with $\mathcal{H}_\rho$ is denoted by H^u.

Theorem 4.1.2 has an obvious corollary concerning the existence of vector fields with invariant tori in N-parameter families, where N is the dimension of H^u. In particular, any analytic family $f : B \cap H^u \to \mathcal{H}_\rho$, sufficiently close to the family $f_0(s) = K + s$, intersects the manifold $\mathcal{W} \cap \mathcal{H}_\rho$ transversally, and Theorem 4.1.2

yields an invariant torus Γ_X for the vector field $X = f(s)$ in the intersection.

If we are just looking for families containing a vector field with frequency vector parallel (but not necessarily equal) to ω, then the number of necessary parameters is reduced by one. A further reduction is possible for vector fields that satisfy a non-degeneracy condition, so that some directions in A^u can be generated via translations. To be more precise, let V be some proper linear subspace of $\mathbb{C}^\ell$. Let $r > \rho > 0$, and let $Z = Z(x, y)$ be a real vector field in $\mathcal{H}_r$ that does not depend on the coordinate x, and that satisfies $\mathbb{P}Z = 0$. Given $\varepsilon > 0$, define

$$g_\varepsilon(z, v) = zK + \varepsilon \mathbb{P} J_v^* Z\,, \qquad z \in \mathbb{C}, \quad v \in V\,, \tag{4.1.4}$$

where $J_v(x, y) = (x, y + v)$. We assume that g_ε is non-degenerate, in the sense that $Dg_\varepsilon(0)$ is one-to-one. Let H_0^u be a linear subspace of H^u that is transversal to the range of $Dg_\varepsilon(0)$, and define

$$f_\varepsilon(s) = K + \varepsilon Z + s\,, \qquad s \in H_0^u\,. \tag{4.1.5}$$

We will see later that $f_\varepsilon(0)$ belongs to $\mathcal{W}$, for small $\varepsilon > 0$.

Given an open neighborhood b in some complex Banach space, denote by $\mathcal{F}(b)$ the space of all bounded analytic functions $f : b \to \mathcal{H}_r$, equipped with the sup-norm.

Lemma 4.1.3 *If $\varepsilon > 0$ is chosen sufficiently small, and if g_ε is non-degenerate, then, given an open neighborhood b_2 of the origin in H_0^u, there exists an open neighborhood B_2 of f_ε in $\mathcal{F}(b_2)$, such that the following holds. For every family $f \in B_2$, we can find a parameter value $s_f \in b_2$, and a nonzero $c_f \in \mathbb{C}$, such that $X = c_f f(s_f)$ belongs to $\mathcal{W}$, and, thus, has an invariant torus $\Gamma_X \in \mathcal{A}_\delta^0$ with* *rotation vector ω. The maps $f \mapsto (c_f, s_f)$ and $f \mapsto \Gamma_X$ are real analytic on B_2.*

This lemma includes cases where $Dg_\varepsilon(0)$ is onto and thus H_0^u is trivial. In such a case, every vector field near $K + \varepsilon Z$ has an invariant torus with frequency vector ω. Consider for example the case of Hamiltonian vector fields, or reversible vector fields with $\ell = d$. Then H^u is of dimension $\ell = d$. Taking for V some $(\ell-1)$-dimensional subspace of $\mathbb{C}^\ell$ not containing ω, it is easy to write down examples (see below) of vector fields $Z \in \mathbb{P}\mathcal{H}_\rho$ for which g_ε is non-degenerate and thus $H_0^u = \{0\}$. Hamiltonian vector fields of this type are also called isoenergetically non-degenerate.

The following example covers different classes of vector fields.

Example 4.1.4 *Consider a basis $\{w_1, w_2, \ldots, w_d\}$ for $\mathbb{R}^d$, with $w_d = \omega$. Let k be the minimum of $d-1$ and ℓ. Define $X_j(x,y) = (y_j w_j, 0)$ for $1 \le j \le k$, and if $k < \ell$, define*

$$X_j(x,y) = \bigl(0, (y_j - y_\ell)^2(e_j + e_\ell)\bigr), \qquad X_\ell(x,y) = \bigl(0, (y_1^2 + \ldots + y_{\ell-1}^2)e_\ell\bigr),$$

for $k < j < \ell$, where $\{e_1, e_2, \ldots, e_\ell\}$ denotes the standard basis for $\mathbb{R}^\ell$. Consider now a real vector field $Z = c_1 X_1 + \ldots + c_\ell X_\ell$, with $c_j \neq 0$ if and only if X_j belongs to $\mathcal{H}_\rho$.

In the case of general vector fields, $c_j \neq 0$ for all j, and we can choose $V = \mathbb{C}^\ell$. The resulting function g_ε is non-degenerate, and the parameter space H_0^u used in Lemma 4.1.3 is of dimension $d + \ell^2 - 1$. The same choice of Z and V can also be used for divergence free vector fields. In this case, H_0^u has dimension $d + \ell^2 - 2$. For reversible or Hamiltonian vector fields, $c_j = 0$ for $j > k$. Taking V to be the span of $\{e_1, \ldots, e_k\}$, we get again a non-degenerate function g_ε, and the parameter space H_0^u is of dimension $d - 1 - k$. In particular, if $k = d-1$, then we are in the situation described above, where every vector field near $K + \varepsilon Z$ has an invariant torus with frequency vector ω.

This chapter is organized as follows. In Section 4.2 we define the spaces of vector fields that we consider. In Section 4.3, we introduce the definition of resonant and nonresonant modes of a vector field. In Section 4.4, we prove a general normal form theorem that can be applied to the problem of elimination of non-resonant modes of a vector field. Section 4.5 contains the properties of a single renormalization group transformation. The construction and composition of a sequence of such transformations $\mathcal{R}_n$, $n \in \mathbb{N}$, according to a multidimensional continued fractions expansion, will be described in Section 4.6. A general stable manifold theorem that applies to such a sequence of transformations is proved in Section 4.7. Section 4.8 is devoted to the application of the renormalization scheme to construction of analytic Diophantine invariant tori in near-integrable systems.

4.2 Spaces of vector fields

In this section, we define precisely the spaces of vector fields that we consider. On the spaces $\mathbb{C}^m$, with $m \in \mathbb{N}_0$, we use the ℓ^∞-norm and ℓ^1-norm that are denoted by $\|\cdot\|$ and $|\cdot|$, respectively. More precisely, for an arbitrary vector $v = (v_1, \ldots, v_m) \in \mathbb{C}^m$, we have $\|v\| = \max_j |v_j|$ and $|v| = \sum_j |v_j|$. For linear operators between normed linear spaces we use the operator, unless stated otherwise.

The considered vector fields are functions on $\mathbb{T}^d \times \mathbb{R}^\ell$, where $\mathbb{T}^d = \mathbb{R}^d/\mathcal{Z}$ and $\mathcal{Z}$ is an arbitrary simple lattice in $\mathbb{R}^d$. Such functions can be lifted to analytic functions on D_ρ, invariant under torus translations $x \mapsto x + z$, with $z \in \mathcal{Z}$. The dual (reciprocal) lattice, i.e. the set of points $\nu \in \mathbb{R}^d$ that satisfy $\exp(i\nu \cdot z) = 1$, for all $z \in \mathcal{Z}$, will be denoted by $\mathcal{V}$. In this chapter, we assume that $\mathcal{Z} = 2\pi\mathbb{Z}^d$, and thus that the frequency lattice $\mathcal{V} = \mathbb{Z}^d$. Here, given $\rho > 0$, define D_ρ to be the set of all points (x, y) in $\mathbb{C}^d \times \mathbb{C}^\ell$ that satisfy $\|\mathrm{Im}\, x\| < \rho$ and $\|y\| < \rho$. If C is a complex Banach space, a bounded analytic function $X : D_\rho \to C$, invariant under the lattice translations, can be expanded in a Fourier-Taylor series as

$$X(x, y) = \sum_{\nu,\kappa} X_{\nu,\kappa} y^\kappa e^{i\nu\cdot x} \,, \tag{4.2.1}$$

where $y^\kappa = \prod_j y_j^{\kappa_j}$ and $\nu \cdot x = \sum_j \nu_j x_j$. Here, the sum ranges over all multi-indices $\nu \in \mathcal{V}$ and $\kappa \in \mathbb{N}_0^\ell$. Define $\mathcal{A}_\rho(C)$ to be the space of all such functions for which the following norm is finite

$$\|X\|_\rho = \sum_{\nu,\kappa} \|X_{\nu,\kappa}\| \rho^{|\kappa|} e^{\rho|\nu|} \,. \tag{4.2.2}$$

If no ambiguity can arise concerning the space C being considered, we will simply write $\mathcal{A}_\rho$ in place of $\mathcal{A}_\rho(C)$.

We denote by $\mathcal{A}'_\rho$ the space of vector fields on D_ρ whose derivatives belong to $\mathcal{A}_\rho$. On this space consider the norm

$$\|X\|'_\rho = \|DX\|_\rho + c\|X\|_\rho \,, \tag{4.2.3}$$

where c is some positive constant, considered fixed from now on. In what follows, we set $c = 1$.

Later on, for the construction of invariant tori, we will also use non-analytic functions, with real domain $D_0 = \mathbb{R}^d \times \{0\}$, 2π-periodic in each of the first d variables. Here, 0 denotes the zero vector in $\mathbb{C}^\ell$. Denote by $\mathcal{A}_0$ the Banach space of continuous functions $f : D_0 \to \mathbb{C}^d \times \mathbb{C}^\ell$, for which the norm $\|f\|_0 = \sum_\nu \|f_\nu\|$ is finite. Here, f_ν are the Fourier coefficients of f. This space can be viewed as the $\rho \to 0$ limit of the spaces $\mathcal{A}_\rho$ defined above.

4.3 Resonant versus non-resonant modes

Unlike the case of Hamiltonians, where any truncation of the Fourier-Taylor series leads a new Hamiltonian, each term in the expansion (4.2.1) of a vector field X is not necessarily a vector field in the same class as the original one. This motivates a new definition of modes of a vector field, that are of the same type as the original vector field.

Denote by $\mathscr{S}$ the generator of the one-parameter group of scalings $\mu \mapsto \mathcal{S}_\mu^*$, that has been defined by (4.1.2).

Definition 4.3.1 *Given any subset J of $I = \mathcal{V} \times \{-1, 0, 1, 2, \ldots\}$, define $P(J)$ to be the joint spectral projection on $\mathcal{A}_\rho$ for the operators $(-i\nabla_x, \mathscr{S})$, associated with the eigenvalues (ν, k) in J. Projection of a vector field associated $P(\{\nu, k\})$, for $(\nu, k) \in I$, will be referred to as a mode of the vector field. The projection $P(\{(0, k)\})$, associated with eigenvalues (0,k), will be denoted by $\mathbb{E}_k$. An averaging operator is defined as $\mathbb{E} = \sum_k \mathbb{E}_k$.*

We will distinguish between two types of modes, "resonant" - which behave well under scaling and "non-resonant" - that can be eliminated by a transformations homotopic to the identity.

Definition 4.3.2 *Given a matrix $T \in \mathrm{SL}(d, \mathbb{Z})$, $\tau > 0$ and $\gamma > 0$, define the resonant index set I^+ to be the set of all pairs $(\nu, k) \in I$ with the property that $|T^\star \nu| \le \tau |\nu|$ or $|T^\star \nu| \le (\tau/\gamma) k$. Define I^- to be the complement of I^+ in I. The projection operators onto the spaces of "resonant" modes and "non-resonant" modes are defined as $\mathbb{I}^+ = P(I^+)$ and $\mathbb{I}^- = P(I^-)$, respectively.*

We assume that $\omega \in \mathbb{R}^d$ is a given vector which satisfies $\|\omega\| = 1$, and that

$$|T^\star \nu_\perp| < \frac{\tau}{2\sqrt{d}}\, |\nu_\perp|\,, \tag{4.3.1}$$

for every vector $\nu_\perp \in \mathbb{R}^d$, perpendicular to ω. The following proposition justifies the terms "resonant" and "non-resonant".

Proposition 4.3.3 *Given a positive number $\sigma < \tau/(2\sqrt{d}\|T\|)$, for every $(\nu, k) \in I^-$, we have*

$$|\omega \cdot \nu| > \sigma|\nu| \qquad \text{and} \qquad |\omega \cdot \nu| > (\sigma/\gamma)k. \tag{4.3.2}$$

Proof: Recall the following inequalities among norms: $\|\nu\| \le \|\nu\|_2 \le |\nu| \le \sqrt{d}\|\nu\|_2 \le d\|\nu\|$, where $\|\nu\|_2$ stands for ℓ^2 norm of a vector $\nu \in \mathbb{R}^d$. These inequalities are used implicitly in the proof together with the equality of matrix norms $|T^\star| = ||T||$. We will also use the decomposition $\nu = \nu_{||} + \nu_\perp$ of a vector ν into a vector $\nu_{||}$, parallel to ω, and a vector $\nu_\perp$, perpendicular to ω.

If (ν, k) is an arbitrary index pair that belongs to I^-, then $|T^\star\nu| > \tau|\nu|$ and $|T^\star\nu| > (\tau/\gamma)k$. By using that $|\nu_\perp| \le \sqrt{d}|\nu|$, we obtain

$$\begin{aligned} \sigma|\nu| &< \frac{\tau}{2\sqrt{d}}\|T\|^{-1}|\nu| \le \frac{1}{\sqrt{d}}\|T\|^{-1}\big(|T^\star\nu| - |T^\star\nu_\perp|\big) \\ &\le \frac{1}{\sqrt{d}}\|T\|^{-1}|T^\star\nu_{||}| \le \frac{1}{\sqrt{d}}|\nu_{||}| \le |\omega\cdot\nu|\,. \end{aligned} \tag{4.3.3}$$

Further, we have

$$\begin{aligned} \frac{\sigma}{\gamma}k &\le \frac{\tau}{2\gamma\sqrt{d}}\|T\|^{-1}k \le \frac{1}{2\sqrt{d}}\|T\|^{-1}|T^\star\nu| \\ &\le \frac{1}{\sqrt{d}}\|T\|^{-1}\big(|T^\star\nu| - |T^\star\nu_\perp|\big) \le |\omega\cdot\nu|\,. \end{aligned} \tag{4.3.4}$$

This completes the proof of the claim. **QED**

Remark 4.3.4 *Alternatively, given $\sigma > 0$ and $\gamma > 0$, we could define I^+ to be the set all pairs $(\nu, k) \in I$ with the property that $|\omega \cdot \nu| < \sigma|\nu|$ or $|\omega \cdot \nu| < (\sigma/\gamma)k$. If $\tau > 0$ has been chosen such that $\sigma < \tau/(2\sqrt{d}\|T\|)$, then every $(\nu, k) \in I^+$ satisfies $|T^\star\nu| \leq \tau|\nu|$ or $|T^\star\nu| \leq (\tau/\gamma)k$.*

Proof: If $|\omega \cdot \nu| \leq \sigma|\nu|$, then

$$\begin{aligned} |T^\star\nu| &\leq |T^\star\nu_{\|}| + |T^\star\nu_\perp| \leq \|T\||\nu_{\|}| + \frac{\tau}{2\sqrt{d}}|\nu_\perp| \\ &\leq \sqrt{d}\|T\|\sigma|\nu| + \frac{\tau}{2}|\nu| \leq \tau|\nu|. \end{aligned} \tag{4.3.5}$$

If $|\omega \cdot \nu| \leq (\sigma/\gamma)k$ and $|T^\star\nu| > \tau|\nu|$, then

$$\begin{aligned} |T^\star\nu| &\leq 2|T^\star\nu| - \tau|\nu| \leq 2|T^\star\nu| - \frac{\tau}{\sqrt{d}}|\nu_\perp| \\ &\leq 2(|T^\star\nu| - |T^\star\nu_\perp|) \leq 2|T^\star\nu_{\|}| \leq (\tau/\gamma)k. \end{aligned} \tag{4.3.6}$$

This statement is proved. **QED**

4.4 A normal form theorem

Here we state and prove a normal form theorem that can be applied to the problem of eliminating nonresonant modes in the renormalization of vector fields on $\mathbb{T}^d \times \mathbb{R}^\ell$. The changes of variables are chosen in such a way as to preserve different classes of vector fields. This aspect will be discussed at the end of this section.

4.4.1 The normal form theorem

Let $D_0 = \mathbb{T}^d \times \{0\}$, where $0 \in \mathbb{R}^\ell$. For every r in some fixed interval $[\rho', \rho]$ of positive real numbers, consider the set D_r of all points $(x, y) \in \mathbb{C}^d \times \mathbb{C}^\ell$ whose (for example

ℓ^∞) distance from (any point in) D_0 is less than r, and let $\mathcal{A}_r$ be a Banach space of vector fields $Y : D_r \to \mathbb{C}^d \times \mathbb{C}^\ell$, that contains the inclusion map I from D_r into $\mathbb{C}^d \times \mathbb{C}^\ell$.

The following is assumed to hold whenever $\rho' \le r < s \le \rho$.

Assumption 4.4.1 *If $X \in \mathcal{A}_s$ and $Z \in \mathcal{A}_r$, then*

(i) $\|X(x,y)\| \le \|X\|_s$ *for all* $(x,y) \in D_s$,

(ii) $(DX)Z \in \mathcal{A}_r$ *and* $\|(DX)Z\|_r \le (s-r)^{-1}\|X\|_s\|Z\|_r$,

(iii) $X \circ (I+Z) \in \mathcal{A}_r$ *and* $\|X \circ (I+Z)\|_r \le \|X\|_s$, *if* $r + \|Z\|_r \le s$.

Let now $\mathbb{I}^-$ be a fixed but arbitrary projection operator on $\mathcal{A}_\rho$, whose restriction to each of the spaces $\mathcal{A}_r$ is a partial isometry on that space. Let K be a fixed vector field in $\mathcal{A}'_\rho$ satisfying $\mathbb{I}^- K = 0$. Given a vector field X near K in $\mathcal{A}'_\rho$, we are looking for a change of variables $\mathcal{U}_X : D_{\rho'} \to D_\rho$, such that the pullback $\mathcal{U}_X^* X$ of X under $\mathcal{U}_X$ belongs to $\mathcal{A}_{\rho'}$ and satisfies

$$\mathbb{I}^- \mathcal{U}_X^* X = 0 \,. \tag{4.4.1}$$

In order to see what conditions may be needed to solve this equation, consider writing $\mathcal{U}_X$ to first order in $X - K$ as the time-one flow Φ_Z^1 for some vector field $Z = \mathbb{I}^- Z$. In this approximation, equation (4.4.1) becomes

$$\mathbb{I}^- (X + [Z, X]) = 0 \,, \tag{4.4.2}$$

where $[Z,X] = (DX)Z - (DZ)X$. This motivates the following condition on K. In addition to $\mathbb{I}^- K = 0$, we assume that the operator $\mathbb{I}^- \hat{K} : \mathbb{I}^- \mathcal{A}'_r \to \mathbb{I}^- \mathcal{A}_r$, defined by

$\hat{K}Z = [K, Z]$, has the inverse which is bounded in norm by a positive constant ζ^{-1}, where $\zeta < 1$, i.e. for every $Z \in \mathbb{I}^{-}\mathcal{A}_r$,

$$\|\hat{K}^{-1}Z\|_r' \leq \zeta^{-1}\|Z\|_r\,, \tag{4.4.3}$$

whenever $\rho' \leq r \leq \rho$. Here, $\|Z\|_r' = \|DZ\|_r + c\|Z\|_r$, with $c > 0$ given, which we consider to be the norm on $\mathcal{A}_r'$.

Our goal in this section is to solve the equation (4.4.1), under the above assumptions. We start with some basic estimates on flows.

The flow $t \mapsto \Phi_X^t$ associated with a vector field $X \in \mathcal{A}_\rho$ is obtained by solving $\frac{d}{dt}\Phi_X^t = X \circ \Phi_X^t$ with initial condition $\Phi_X^0 = \mathrm{I}$. Writing $\Phi_X^t = \mathrm{I} + Y(t)$, this is equivalent to solving the integral equation

$$Y(t) = \int_0^t X \circ [\mathrm{I} + Y(s)]\, ds\,. \tag{4.4.4}$$

In what follows, any reference to a space $\mathcal{A}_\varrho$ implicitly assumes that we have $\rho' \leq \varrho \leq \rho$.

Proposition 4.4.2 *Let ϱ', ϱ and t_0 be positive real numbers and $X \in \mathcal{A}_\varrho$ a given vector field such that we have $\varrho' + t_0\|X\|_\varrho < \varrho$. Then the equation (4.4.4) has a unique continuous solution $t \mapsto Y(t) \in \mathcal{A}_{\varrho'}$ on the interval $|t| \leq t_0$, and*

$$\|\Phi_X^t - \mathrm{I}\|_{\varrho'} \leq \|tX\|_\varrho\,. \tag{4.4.5}$$

Proof: By Assumption 4.4.1 and the contraction mapping principle, the equation (4.4.4) has a unique solution $t \mapsto Y(t) \in \mathcal{A}_{\varrho - r}$, where $\|tX\|_\varrho < r < \varrho - \varrho'$, for t near zero. The solution can be continued as usual, as long as $\|Y(t)\|_{\varrho - r} < r$. But on any interval containing zero, where $\|Y(t)\|_{\varrho-r} < r$, we have by (4.4.4) the bound

$$\|Y(t)\|_{\varrho-r} \leq \|tX\|_{\varrho} \,. \tag{4.4.6}$$

Thus, equation (4.4.4) has a continuous solution Y for all times t satisfying $\|tX\|_{\varrho} < r$. The bound (4.4.5) follows from (4.4.6) and the fact that $Y(t) = \Phi_X^t - I$. **QED**

Proposition 4.4.3 *Let $0 < r < \varrho$ and $t \in \mathbb{R}$. Let Z and X be two vector fields in $\mathcal{A}_{\varrho}$, satisfying $\|tZ\|_{\varrho} \leq r\varepsilon$ and $\|tDZ\|_{\varrho} \leq s\varepsilon$, with $\varepsilon \leq 1/6$, and $s > 0$. Then $(\Phi_Z^t)^* X$ belongs to $\mathcal{A}_{\varrho-r}$, and*

$$\begin{aligned} \left\|(\Phi_Z^t)^* X - X\right\|_{\varrho-r} &\leq 3e^s \|X\|_{\varrho}\, \varepsilon \,, \\ \left\|(\Phi_Z^t)^* X - X - t[Z,X]\right\|_{\varrho-r} &\leq 7e^s \|X\|_{\varrho}\, \varepsilon^2 \,. \end{aligned} \tag{4.4.7}$$

Proof: It suffices to consider only the case $t = 1$, since we can always rescale time. Let n be a fixed positive integer. Using Assumption 4.4.1, and Cauchy's formula with contour $|z| = 1$, to estimate

$$(DX)Z = n\varepsilon \frac{d}{dz}\left[X \circ \left(\mathrm{I} + \frac{z}{n\varepsilon} Z\right)\right]_{z=0} , \tag{4.4.8}$$

we obtain the bound

$$\|\widehat{Z} X\|_{\varrho'-r/n} \leq (n\varepsilon + s\varepsilon)\|X\|_{\varrho} \,, \tag{4.4.9}$$

where $\varrho' = \varrho$. This bound can be iterated n times, with ϱ' decreasing by r/n after each step, and we find

$$\frac{1}{n!}\left\|(\widehat{Z})^n X\right\|_{\varrho-r} \leq \frac{1}{n!}(n+s)^n \varepsilon^n \|X\|_{\varrho} \leq \frac{n^n}{n!} e^s \varepsilon^n \|X\|_{\varrho} \leq \frac{1}{2}(e\varepsilon)^n e^s \|X\|_{\varrho} \,. \tag{4.4.10}$$

In the last inequality, we have used Stirling's formula. Now

$$\left\|(\Phi_Z^1)^* X - X\right\|_{\varrho-r} = \left\|\sum_{n=1}^{\infty} \frac{1}{n!}(\widehat{Z})^n X\right\|_{\varrho-r} \leq \frac{\varepsilon}{2} \cdot \frac{e^{s+1}}{1-e\varepsilon}\|X\|_{\varrho}\,, \tag{4.4.11}$$

and the first bound in (4.4.7) follows. The second bound is obtained analogously, with the sum in (4.4.11) starting at $n = 2$. **QED**

Returning to the main problem, our first step is to solve the equation (4.4.2).

Proposition 4.4.4 *Let $0 < r < \varrho$. If X is a vector field in $\mathcal{A}'_\varrho$, satisfying*

$$\|X - K\|'_\varrho \leq \tfrac{1}{4}\zeta c\,, \qquad \|\mathbb{I}^- X\|_\varrho \leq \tfrac{1}{12}\zeta c r\,, \tag{4.4.12}$$

then the equation (4.4.2) has a unique solution $Z \in \mathbb{I}^- \mathcal{A}'_\varrho$, that satisfies $\|Z\|'_\varrho \leq \frac{2}{\zeta}\|\mathbb{I}^- X\|_\varrho$. Moreover, $(\Phi_Z^1)^* X$ belongs to $\mathcal{A}_{\varrho-r}$ and satisfies

$$\begin{aligned} \left\|(\Phi_Z^1)^* X - X\right\|_{\varrho-r} &\leq \frac{6e^{cr}}{\zeta c r}\|\mathbb{I}^- X\|_\varrho \|X\|_\varrho\,, \\ \left\|(\Phi_Z^1)^* X - X - [Z, X]\right\|_{\varrho-r} &\leq \frac{28e^{cr}}{(\zeta c r)^2}\|\mathbb{I}^- X\|_\varrho^2 \|X\|_\varrho\,. \end{aligned} \tag{4.4.13}$$

Proof: The first condition in (4.4.12) implies that

$$\|[Z, X - K]\|_\varrho \leq \frac{2}{c}\|Z\|'_\varrho \|X - K\|'_\varrho \leq \frac{\zeta}{2}\|Z\|'_\varrho\,, \tag{4.4.14}$$

for every $Z \in \mathcal{A}'_\varrho$. Thus, the operator $\hat{f}$, where $f = X - K$, is bounded from $\mathcal{A}'_\varrho$ into $\mathcal{A}_\varrho$. Writing the equation (4.4.2) in the form $(\mathbb{I} + \mathbb{I}^- \hat{f}(\mathbb{I}^- \hat{K})^{-1})\mathbb{I}^- \hat{K} Z = \mathbb{I}^- X$, we find that for every $Z \in \mathbb{I}^- \mathcal{A}_\varrho$,

$$\|\mathbb{I}^- \hat{f}(\mathbb{I}^- \hat{K})^{-1} Z\|_\varrho \leq \frac{\zeta}{2}\|(\mathbb{I}^- \hat{K})^{-1} Z\|'_\varrho \leq \frac{1}{2}\|Z\|_\varrho. \tag{4.4.15}$$

Thus, the operator $\mathbb{I}^{-}\hat{f}(\mathbb{I}^{-}\hat{K})^{-1}\mathbb{I}^{-}$ is bounded on $\mathcal{A}_{\varrho}$, with the operator norm smaller than or equal to 1/2. Equation (4.4.2) can then be solved by inverting the operator $\mathbb{I}+\mathbb{I}^{-}\hat{f}(\mathbb{I}^{-}\hat{K})^{-1}$ by means of a Neumann series. The solution $Z \in \mathbb{I}^{-}\mathcal{A}'_{\varrho}$ satisfies the desired bound.

The bounds (4.4.13) follow from Proposition 4.4.3, setting $\varepsilon = (2/\zeta c r)\|\mathbb{I}^{-}X\|_{\varrho}$ and $s = cr$. **QED**

Our next step is to iterate the map $X \mapsto (\Phi^1_Z)^* X$ described in Proposition 4.4.4, by starting with a vector field $X = X_0$ and setting

$$X_{n+1} = (\Phi^1_{Z_n})^* X_n\,, \qquad \mathbb{I}^{-}(X_n + [Z_n, X_n]) = 0\,, \tag{4.4.16}$$

for $n \in \mathbb{N}_0$. The expectation is that the maps

$$U_n = \Phi^1_{Z_0} \circ \Phi^1_{Z_1} \circ \ldots \circ \Phi^1_{Z_{n-1}} \tag{4.4.17}$$

converge to a solution $\mathcal{U}_X$ of the equation (4.4.1), as n tends to infinity. This leads to the main result of this section.

Let in the following $r = \rho - \rho'$. Choose $R \geq \|K\|_{\rho} + \zeta c$ and $\varepsilon > 0$, subject to the constraints

$$\varepsilon \leq 2^{-6}\zeta c r\,, \qquad \varepsilon \leq 2^{-12}\zeta^2 c^2 r^2 e^{-cr}(1+c)^{-1}(1+r)^{-1}R^{-1}\,. \tag{4.4.18}$$

Theorem 4.4.5 (Normal form) *If X is a vector field in $\mathcal{A}'_{\rho}$ such that*

$$\|X - K\|'_{\rho} \leq 2^{-3}\zeta c\,, \qquad \|\mathbb{I}^{-}X\|_{\rho} \leq \varepsilon\,, \tag{4.4.19}$$

with $\varepsilon > 0$ satisfying conditions (4.4.18)*, then there exists an analytic change* *of coordinates $\mathcal{U}_X : D_{\rho'} \to D_\rho$, such that $\mathcal{U}_X^* X$ belongs to $\mathcal{A}_{\rho'}$ and satisfies equation* (4.4.1). *The map $X \mapsto \mathcal{U}_X - \mathrm{I}$ takes values in $\mathcal{A}_{\rho'}$, is continuous in the region defined by inequalities* (4.4.19), *analytic in the interior of this region, and satisfies the bounds*

$$
\begin{aligned}
\|\mathcal{U}_X - \mathrm{I}\|_{\rho'} &\leq \frac{3}{\zeta c}\|\mathbb{I}^- X\|_\rho\,, \\
\|\mathcal{U}_X^* X - X\|_{\rho'} &\leq 32R\frac{e^{cr}}{\zeta cr}\|\mathbb{I}^- X\|_\rho\,, \\
\big\|\mathcal{U}_X^* X - X - [Z, X]\big\|_{\rho'} &\leq \Big(2^{11}\frac{e^{cr}}{\zeta r} + 1\Big)28R\frac{e^{cr}}{(\zeta cr)^2}\|\mathbb{I}^- X\|_\rho^2\,.
\end{aligned}
\tag{4.4.20}
$$

Here, $Z \in \mathbb{I}^- \mathcal{A}'_\rho$ is defined by (4.4.2) *and satisfies the bound $\|Z\|'_\rho \leq \frac{2}{\zeta}\|\mathbb{I}^- X\|_\rho$.*

Proof: Let $\rho_0 = \rho$ and $\rho_{m+1} = \rho_m - 2r_m$, where $r_m = 2^{-m-2}r$, for $m \in \mathbb{N}_0$. Our first goal is to prove that (4.4.16) defines a sequence of vector fields $X_m \in \mathcal{A}'_{\rho_m}$, satisfying

$$
\|X_m - X_{m-1}\|'_{\rho_m} \leq 2^{-m-3}\zeta c\,, \qquad \|\mathbb{I}^- X_m\|_{\rho_m} \leq 8^{-m}\varepsilon\,. \tag{4.4.21}
$$

If we define $X_{-1} = K$ and $X_0 = X$, then these bounds hold for $m = 0$ by (4.4.19). Assume now that (4.4.21) holds for all $m \in \mathbb{N}_0$ with $m \leq n$. Then, by summing up the bounds on $X_m - X_{m-1}$ for $m \leq n$, we obtain the first inequality in

$$
\|X_n - K\|'_{\rho_n} \leq \frac{1}{4}\zeta c\,, \qquad \|\mathbb{I}^- X_n\|_{\rho_n} \leq 4^{-n-2}\zeta c r_n\,. \tag{4.4.22}
$$

The second inequality follows from (4.4.21), by substituting the first bound in (4.4.18) on ε. Thus, Proposition 4.4.4 guarantees a unique solution to (4.4.16), and yields the bounds

$$\|X_{n+1}-X_n\|_{\rho_n-r_n} \leq 6\frac{e^{cr}}{\zeta cr}4^{-n+1}R\varepsilon\,, \qquad \|\mathbb{I}^- X_{n+1}\|_{\rho_n-r_n} \leq 7\frac{e^{cr}}{(\zeta cr)^2}4^{-2n+3}R\varepsilon^2\,. \tag{4.4.23}$$

Here, we have used also that $\|X_n\|_{\rho_n} \leq R$, which follows from the first inequality in (4.4.22). By using the second condition in (4.4.18), together with the fact that $\|F\|'_{\rho_n-2r_n} \leq r_n^{-1}\|F\|_{\rho_n-r_n}$, we now obtain (4.4.21) for $m = n+1$ from the bounds (4.4.23).

Next, consider the functions $\phi_j = \Phi^1_{Z_j} - \mathrm{I}$. By Proposition 4.4.2 and Proposition 4.4.4,

$$\|\phi_j\|_{\rho_{j+1}} \leq \|Z_j\|_{\rho_j} \leq \frac{2}{\zeta c}\|\mathbb{I}^- X_j\|_{\rho_j} < r_j\,. \tag{4.4.24}$$

This shows that $U_{m,n} = \Phi^1_{Z_m} \circ \Phi^1_{Z_{m+1}} \circ \ldots \circ \Phi^1_{Z_{n-1}}$ defines a function in $\mathrm{I} + \mathcal{A}_{\rho_n}$ that takes values in D_{ρ_m}. Here, and in what follows, it is assumed that $0 \leq m < n$. Setting $U_{j,j} = \mathrm{I}$, we have the bound

$$\|U_n - U_m\|_{\rho'} = \left\|\sum_{j=m}^{n-1} \phi_j \circ U_{j+1,n}\right\|_{\rho'} \leq \sum_{j=m}^{n-1} \|\phi_j\|_{\rho_{j+1}} \leq \sum_{j=m}^{n-1} \frac{2}{\zeta c}8^{-j}\varepsilon\,. \tag{4.4.25}$$

This shows that $n \mapsto U_n$ converges in $\mathrm{I} + \mathcal{A}_{\rho'}$ to a limit $\mathcal{U}_X$ that takes values in D_ρ, and that satisfies the first inequality in (4.4.20), if we set $\varepsilon = \|\mathbb{I}^- X\|_\rho$. Clearly, $X_n \to \mathcal{U}_X^* X$ in $\mathcal{A}_{\rho'}$. The second inequality in (4.4.20) is now obtained by using the first bound in (4.4.23).

Since $\mathcal{U}_X^* X = \mathcal{U}_{X_1}^* X_1$ with $X_1 = (\Phi^1_Z)^* X$, we have

$$\mathcal{U}_X^* X - X - [Z,X] = \left(\mathcal{U}_{X_1}^* X_1 - X_1\right) + \left((\Phi^1_Z)^* X - X - [Z,X]\right). \tag{4.4.26}$$

The term in the first parenthesis of (4.4.26) can be estimated in the same way as

$\|\mathcal{U}_X^* X - X\|_{\rho'}$, yielding the bound

$$\left\|\mathcal{U}_{X_1}^* X_1 - X_1\right\|_{\rho'} \leq 32R\frac{e^{cr}}{\zeta cr}\|\mathbb{I}^- X_1\|_{\rho_1}. \tag{4.4.27}$$

Since $\|\mathbb{I}^- X_1\|_{\rho_1} \leq \|(\Phi_Z^1)^* X - X - [Z, X]\|_{\rho_1}$, the third bound in (4.4.20) now follows from the second inequality in (4.4.13).

The analyticity of the map $X \mapsto \mathcal{U}_X$ follows from the uniform convergence of $U_n \to \mathcal{U}_X$. **QED**

4.4.2 Class-preserving property of the elimination map

We conclude this section with a discussion of invariance properties of the map $X \to \mathcal{U}_X^* X$. In the following, we will assume that the vector field K belongs to a considered class of vector fields, and that the projection operator $\mathbb{I}^-$ preserves that class of vector fields. In particular, for the vector field $K = (\omega, 0)$, the classes of vector fields mentioned at the beginning of this chapter, and the projection operator $\mathbb{I}^-$ from Definition 4.3.2, acting on spaces $\mathcal{A}_r$ defined in Section 4.2, this is certainly the case.

Now consider classes of vector fields that, with the above defined commutator, form Lie subalgebras of vector fields, e.g. Hamiltonian and divergence-free vector fields and vector fields symmetric with respect to the involution $G : (x, y) \mapsto (x, -y)$. The entire analysis of this section could then be restricted to such a class of vector fields. Since the solution of (4.4.2) is unique, we find that the solution Z belongs to the same class as X. This can be also seen, more explicitly, from the Neumann series form of the solution. As a consequence, for example, the change of coordinates $\mathcal{U}_X$ is symplectic, whenever X is Hamiltonian, volume-preserving, whenever X is divergence-free, and commutes with the involution G, whenever X

is symmetric with respect to it. This includes vector fields that are Hamiltonian with respect to the pullback of the standard symplectic form under linear transformations $\mathcal{C}(x, y) = (x, Cy)$, where C can be any real nonsingular $\ell \times \ell$ matrix. This follows from the fact that $\mathcal{C}^*$ commutes with $\mathbb{I}^-$.

Next, consider the class of vector fields reversible with respect to a linear map G on $\mathbb{C}^d \times \mathbb{C}^\ell$ which is an involution and leaves the domains D_r invariant. Assume that G^* is an isometry on each of the spaces $\mathcal{A}_r$, and that it commutes with $\mathbb{I}^-$. An example of such a map is $G(x, y) = (-x, y)$. Since the commutator of a symmetric and reversible vector field is reversible (all with respect to G), we see that if X is reversible, the operator $\mathbb{I}^- \hat{X}$ maps the symmetric subspace of $\mathbb{I}^- \mathcal{A}'_\rho$ to the reversible subspace of $\mathbb{I}^- \mathcal{A}_\rho$. As the proof of Proposition 4.4.4 shows, this operator has (under the given assumptions) a bounded inverse, so the solution Z of equation (4.4.2) is symmetric. Consequently, at each elimination step the flow map Φ_Z^1 commutes with G, and the same is true for $\mathcal{U}_X$.

4.5 A single renormalization transformation $\mathcal{R}$

In this section we construct a single renormalization step. Apart from elimination of non-resonant modes, a single renormalization transformation involves the pullback of a phase-space scaling. This scaling is constructed by using a matrix $T \in \mathrm{SL}(d, \mathbb{Z})$, that is assumed to be given in this section. In addition, positive constants $\rho' < \rho < \varrho$ are assumed to be given and fixed from now on.

4.5.1 Scaling and analyticity improving

Let ω be a ℓ^∞-unit vector in $\mathbb{R}^d$, let $K = (\omega, 0)$, and let T and $\bar{T}$ be two matrices in $\mathrm{SL}(d, \mathbb{R})$, which satisfy

$$\max\{\|T\|, \|T^{-1}\|, \|\bar{T}\|, \|\bar{T}^{-1}\|\} < \tau/(2\sqrt{d}\sigma)\,. \tag{4.5.1}$$

We will choose $\bar{T} = (T^\star)^{-1}$ if $d = \ell$, or $\bar{T} = I$, otherwise. By this choice, the condition (4.5.1) becomes a constraint on T only. We assume that there exist $\gamma \geq 1$ and positive constants $\mu, \sigma, \tau < 1$ satisfying the following assumptions.

Assume that the following conditions are satisfied,

$$\tau \leq \rho'/(2\varrho)\,, \qquad b\mu < \hat{\mu}\,, \qquad \tau \ln(\hat{\mu}/\mu) \leq \frac{\rho'}{2(\gamma+1)}\,. \tag{4.5.2}$$

Here $b = \exp(\rho'/2)$ and $\hat{\mu} = (\rho'/\varrho)(\sigma/\tau)$.

At each renormalization step scaling of the phase space is performed with the linear map $\mathcal{T}_\mu = S_\mu \circ \mathcal{T}$, defined by (4.1.2). Under the pullback of this scaling, a vector field X transform as $\mathcal{T}^*\mathcal{S}^*_\mu(X)$. When restricting its domain to the space of resonant vector fields, this operator is analyticity improving.

Lemma 4.5.1 *Let $0 < \rho' < \varrho$ be fixed and assume that the relevant constants have been chosen such that condition (4.5.2) is satisfied. Then $\mathcal{T}^*\mathcal{S}^*_\mu$ defines a bounded linear operator from $\mathbb{I}^+\mathcal{A}_{\rho'}$ to $\mathcal{A}_\varrho$, satisfying*

$$\begin{aligned} \|\mathcal{T}^*\mathcal{S}^*_\mu \mathbb{E}_k X\|_\varrho &\leq N(T)(\mu/\hat{\mu})^k \|\mathbb{E}_k X\|_{\rho'} \\ \|\mathcal{T}^*\mathcal{S}^*_\mu \mathbb{I}^+(\mathbb{I}-\mathbb{E})X\|_\varrho &\leq N(T)(b\mu/\hat{\mu})^\gamma \|\mathbb{I}^+(\mathbb{I}-\mathbb{E})X\|_{\rho'}\,, \end{aligned} \tag{4.5.3}$$

where $N(T) = \|T^{-1}\| + (\varrho/\rho')\|\bar{T}^{-1}\|\|\bar{T}\|$.

Proof: By our choice of norm (4.2.2), it suffices to verify the given bounds for vector fields $X = P(J)Y$, with J containing a single point. Let

$$J = \{(\nu, k)\}\,, \qquad A = \varrho|T^\star\nu| - \rho'|\nu| + k\ln(\mu/\hat{\mu})\,. \tag{4.5.4}$$

Then, it follows essentially from the definitions that

$$\|\mathcal{T}^*\mathcal{S}_\mu^* P(J)Y\|_\varrho \le N(T)e^A\|P(J)Y\|_{\rho'}\,. \tag{4.5.5}$$

Setting $\nu = 0$ yields the first bound in (4.5.3).

In order to prove the second bound, assume that (ν, k) belongs to I^+, and that $\nu \neq 0$. Consider first the case $|T^\star\nu| \le \tau|\nu|$. Then $|\nu| \ge \tau^{-1}$, and we obtain

$$\begin{aligned} A &\le (\varrho\tau - \rho')|\nu| + k\ln(\mu/\hat{\mu}) \le \left(\varrho - \frac{\rho'}{\tau}\right) - \ln(\mu/\hat{\mu}) \\ &\le -\frac{\rho'}{2\tau} - \ln(\mu/\hat{\mu}) \le \gamma\ln(\mu/\hat{\mu})\,. \end{aligned} \tag{4.5.6}$$

In the last inequality we have used condition (4.5.2).

Now consider the case $\tau|\nu| < |T^\star\nu| \le (\tau/\gamma)k$. By using that $\varrho\tau/\gamma \le \rho'/(2\gamma)$, and $k > \gamma$, we find that

$$A \le \varrho\frac{\tau}{\gamma}k + k\ln(\mu/\hat{\mu}) \le k\ln(b\mu/\hat{\mu}) \le \gamma\ln(b\mu/\hat{\mu})\,. \tag{4.5.7}$$

The second bound in (4.5.3) now follows from (4.5.6) and (4.5.7). **QED**

4.5.2 Elimination of non-resonant modes

Here we prove that given a vector field $X \in \mathcal{A}'_\rho$ sufficiently close to $K = (\omega, 0)$, one can construct a phase space coordinate change $\mathcal{U}_X : D_{\rho'} \to D_\rho$, with $0 < \rho' < \rho$, that eliminates the non-resonant modes of X, i.e. such that the transformed vector field $\mathcal{U}_X^* X$ belongs to $\mathcal{A}_{\rho'}$ and satisfies

$$\mathbb{I}^- \mathcal{U}_X^* X = 0. \tag{4.5.8}$$

Moreover, we construct that coordinate change in such a way that the map $X \mapsto \mathcal{U}_X^* X$ preserves the class of the vector field.

For that reason, we verify the assumptions of the normal form theorem of Section 4.4. In the considered setting, it is a trivial exercise to verify the Assumption 4.4.1.

Proposition 4.5.2 *If $X \in \mathcal{A}_s$ and $Z \in \mathcal{A}_r$, then*

(i) $\|X(x, y)\| \leq \|X\|_s$, *for all* $(x, y) \in D_s$,

(ii) $(DX)Z \in \mathcal{A}_r$ *and* $\|(DX)Z\|_r \leq (s - r)^{-1}\|X\|_s\|Z\|_r$,

(iii) $X \circ (I + Z) \in \mathcal{A}_r$ *and* $\|X \circ (I + Z)\|_r \leq \|X\|_s$, *if* $r + \|Z\|_r \leq s$.

The next proposition verifies the boundness of the inverse of the operator $\mathbb{I}^- \hat{K}$, where $\hat{K}Z = [K, Z]$, for any $Z \in \mathcal{A}_r$. Here and in what follows $\rho' \leq r \leq \rho$.

We assume that the matrix $T^\star$ is strongly contracting on the orthogonal complement of ω, with the contraction factor of smaller than $\tau/2\sqrt{d}$, i.e. that it satisfies the condition (4.3.1). We recall that this condition, together with the condition on a positive σ,

$$\sigma < \tau/(2\sqrt{d}\|T\|)\,, \tag{4.5.9}$$

leads to the bounds on small devisors obtained in Proposition 4.3.3.

Proposition 4.5.3 *If the condition $\sigma > 0$ satisfies the condition* (4.5.9)*, then $\hat{K}^{-1}$ is a bounded linear operator from $\mathbb{I}^{-}\mathcal{A}_r$ into $\mathbb{I}^{-}\mathcal{A}'_r$, with the operator norm satisfying*

$$\|\hat{K}^{-1}\|'_r \leq \frac{r(c+1)+\gamma+\ell}{r\sigma} \tag{4.5.10}$$

Proof: Consider the action of the operator $\hat{K}^{-1}$ on a mode

$$X_{\nu,k}(x,y) = X_{\nu,k}(y)e^{ix\cdot\nu}, \tag{4.5.11}$$

with $(\nu,k) \in I^{-}$, i.e.

$$\hat{K}^{-1}X_{\nu,k}(x,y) = \frac{1}{i\omega\cdot\nu}X_{\nu,k}(y)e^{ix\cdot\nu}. \tag{4.5.12}$$

Since $\|\hat{K}^{-1}X_{\nu,k}\|_r = \|X_{\nu,k}\|_r/|\omega\cdot\nu|$, and

$$\sum_{j=1}^{d}\left\|\frac{\partial}{\partial x_j}\hat{K}^{-1}X_{\nu,k}\right\|_r \leq \frac{|\nu|}{|\omega\cdot\nu|}\|X_{\nu,k}\|_r, \quad \sum_{j=1}^{\ell}\left\|\frac{\partial}{\partial y_j}\hat{K}^{-1}X_{\nu,k}\right\|_r \leq \frac{k+\ell}{r|\omega\cdot\nu|}\|X_{\nu,k}\|_r, \tag{4.5.13}$$

using Proposition 4.3.3, we find that, as an operator from $\mathbb{I}^{-}\mathcal{A}_r$ into $\mathbb{I}^{-}\mathcal{A}_r$, $\hat{K}^{-1}$ satisfies the following bounds

$$\|\hat{K}^{-1}\|_r \leq \frac{1}{\sigma}, \qquad \sum_{j=1}^{d}\left\|\frac{\partial}{\partial x_j}\hat{K}^{-1}\right\|_r \leq \frac{1}{\sigma}, \qquad \sum_{j=1}^{\ell}\left\|\frac{\partial}{\partial y_j}\hat{K}^{-1}\right\|_r \leq \frac{\gamma+\ell}{\sigma r}. \tag{4.5.14}$$

This implies the bound (4.5.10). **QED**

This proposition verifies the main assumption of Theorem 4.4.5 in Section 4.4 with

$$\zeta = \inf_r\left\{\frac{r\sigma}{r(c+1)+\gamma+\ell}\right\}. \tag{4.5.15}$$

The implications of this theorem can now be summarized in the following lemma.

Lemma 4.5.4 *Under the above assumptions, there exist positive constants C and C', such that the following holds. Denote by $\mathcal{D}$ the open ball in $\mathcal{A}_\varrho$ of radius $\varepsilon = C(\sigma/\gamma)^2$, centered at K. Then for every $X \in \mathcal{D}$, there exists an analytic change of coordinates $\mathcal{U}_X : D_{\rho'} \to D_\rho$, such that $\mathcal{U}_X^* X$ belongs to $\mathcal{A}_{\rho'}$ and satisfies equation* (4.5.8)*. The map $X \mapsto \mathcal{U}_X$ is analytic from $\mathcal{D}$ to the affine space $\mathrm{I} + \mathcal{A}_{\rho'}$ and satisfies the bounds in Theorem 4.4.5, with $\zeta = C'\sigma/\gamma$.*

4.5.3 A one-step renormalization transformation $\mathcal{R}$

Now we are ready to construct the renormalization operator. We assume that a ℓ^∞-unit vector is given and that the matrix $T \in \mathrm{SL}(d, \mathbb{Z})$ is chosen such that the conditions (4.3.1) and (4.5.1) are satisfied. In addition, we assume that positive constants $\gamma \geq 1$, and $\tau, \sigma, \mu < 1$, have been chosen such that the constraints (4.5.2) and (4.5.9) are valid. In what follows, a quantity will be called *universal* if it is independent of the choice of T, ω, γ, τ, σ, and μ.

Thus, by Lemma 4.5.4, there exists a universal constant $R > 0$ such that the map $X \mapsto \mathcal{U}_X$ is bounded and analytic on an open ball in $\mathcal{A}_\varrho$ of radius $2R(\sigma/\gamma)^2$, centered at $K = (\omega, 0)$. On this ball, we can now define our RG transformation $\mathcal{R}$ as in (4.1.3). The normalization constant η is defined as $\eta = \|T^{-1}\omega\|$, so that $\tilde{\omega} := \mathcal{R}(\omega) = \eta^{-1} T^{-1} \omega$ is again a unit vector. Notice that, by construction, we have $\mathcal{U}_X = \mathrm{I}$, whenever X is resonant. Thus, $\mathcal{R} \circ \mathbb{I}^+$ is linear, and so is $\mathcal{R} \circ \mathbb{E}$.

Let $\mathbb{P} = \mathbb{E}_{-1} + \mathbb{E}_0$. The subspace $\mathbb{P}\mathcal{A}_\varrho$ consists of all vector fields of the form $X(x, y) = (u, My + v)$, where $u \in \mathbb{C}^d$, $v \in \mathbb{C}^\ell$, and $M \in \mathbb{C}^{\ell \times \ell}$ are arbitrary. This subspace is invariant under $\mathcal{R}$. The restriction of $\mathcal{R}$ to this subspace, which is linear, will be denoted by $\mathcal{L}$.

We obtain the following bounds on a single renormalization transformation.

Theorem 4.5.5 *There exist universal constants $R, C_0 > 0$, such that the following holds. Let $\mathcal{D}$ be the open ball in $\mathcal{A}_\varrho$ of radius $2R(\sigma/\gamma)^2$, centered at K. Then $\mathcal{R}$ is bounded and analytic on $\mathcal{D}$, satisfying*

$$\begin{aligned}
\|(\mathbb{I}-\mathbb{E})\mathcal{R}(X)\|_\varrho &\leq C_0\eta^{-1}(\gamma/\sigma)(C_0\tau/\sigma)^{\gamma+2}\mu^\gamma\|(\mathbb{I}-\mathbb{E})X\|_\varrho\,, \\
\|(\mathbb{I}-\mathbb{P})\mathcal{R}(X)\|_\varrho &\leq C_0\eta^{-1}(\gamma/\sigma)(\tau/\sigma)^3\mu\|(\mathbb{I}-\mathbb{P})X\|_\varrho\,, \\
\|\mathbb{E}\mathcal{R}(X)-\mathcal{R}(\mathbb{E}X)\|_\varrho &\leq C_0\eta^{-1}(\gamma/\sigma)^3(\tau/\sigma)\mu^{-1}\|(\mathbb{I}-\mathbb{E})X\|_\varrho^2\,, \\
\|\mathcal{L}^{-1}\| &\leq C_0\eta(\tau/\sigma)^2\,.
\end{aligned} \tag{4.5.16}$$

Proof: Let R be half the constant C from Lemma 4.5.4, so that we can apply the estimates from Theorem 4.4.5. Let X be an arbitrary vector field in $\mathcal{D}$.

From Lemma 4.5.1, we obtain

$$\begin{aligned}
\|(\mathbb{I}-\mathbb{E})\mathcal{R}(X)\|_\varrho &= \eta^{-1}\|\mathcal{T}^*\mathcal{S}_\mu^*(\mathbb{I}-\mathbb{E})\mathcal{U}_X^*X\|_\varrho \\
&\leq C_1\eta^{-1}(c\tau/\sigma)^{\gamma+2}\mu^\gamma\left[\|(\mathbb{I}-\mathbb{E})X\|_{\rho'}+\|\mathcal{U}_X^*X-X\|_{\rho'}\right],
\end{aligned} \tag{4.5.17}$$

for $c=\exp(\rho'/2)\varrho/\rho'$ and some constant $C_1>0$. Here and in what follows C_n, $n\in\mathbb{N}$, denote positive universal constants. Using the bound (4.4.20) on the norm of $\mathcal{U}_X^*X-X$, together with the fact that $\mathbb{I}^- = \mathbb{I}^-(\mathbb{I}-\mathbb{E})$, we obtain the first inequality in (4.5.16).

Similarly, Lemma 4.5.1 implies that

$$\|\mathbb{E}_k\mathcal{R}(X)\|_\varrho \leq C_2\eta^{-1}(\tau/\sigma)^3\mu\left[\|\mathbb{E}_kX\|_{\rho'}+\|\mathbb{E}_k(\mathcal{U}_X^*X-X)\|_{\rho'}\right], \tag{4.5.18}$$

for all $k\geq 1$. Summing over $k\geq 1$ to get a bound on $\|(\mathbb{E}-\mathbb{P})\mathcal{R}(X)\|_\varrho$, and then adding (4.5.17), yields a bound analogous to (4.5.18), but with $\mathbb{E}_k$ replaced by

$\mathbb{I}-\mathbb{P}$. Applying again the bound (4.4.20) on the norm of $\mathcal{U}_X^* X - X$, we obtain the second inequality in (4.5.16).

By Lemma 4.5.1, we also have the bound

$$\begin{aligned} \|\mathbb{E}\mathcal{R}(X) - \mathcal{R}(\mathbb{E}X)\|_{\varrho} &= \eta^{-1}\|\mathcal{T}^*\mathcal{S}_\mu^*\mathbb{E}(\mathcal{U}_X^* X - X)\|_{\varrho} \\ &\leq C_3\eta^{-1}(\tau/\sigma)\mu^{-1}\|\mathbb{E}(\mathcal{U}_X^* X - X)\|_{\rho'}\,. \end{aligned} \tag{4.5.19}$$

Using the third of the bounds (4.4.20) in Theorem 4.4.5, the norm on the right hand side of this inequality can be estimated by

$$\|\mathbb{E}(\mathcal{U}_X^* X - X)\|_{\rho'} \leq C_4(\gamma/\sigma)^3\|(\mathbb{I}-\mathbb{E})X\|_{\rho}^2 + \|\mathbb{E}[Z, X]\|_{\rho'}\,, \tag{4.5.20}$$

where $Z = \mathbb{I}^- Z$ is the solution of equation $\mathbb{I}^-(X + [Z, X]) = 0$. Since $\mathbb{E}Z = 0$, we have $\mathbb{E}[Z, \mathbb{E}X] = 0$. Thus,

$$\begin{aligned} \|\mathbb{E}[Z, X]\|_{\rho'} &= \|\mathbb{E}[Z, (\mathbb{I}-\mathbb{E})X]\|_{\rho'} \\ &\leq C_5\|Z\|_{\rho}\|(\mathbb{I}-\mathbb{E})X\|_{\rho} \leq C_6(\gamma/\sigma)\|(\mathbb{I}-\mathbb{E})X\|_{\rho}^2\,. \end{aligned} \tag{4.5.21}$$

In the last step, we have used the bound on $\|Z\|_{\rho}$ from Proposition 4.4.4. Combining the last three inequalities yields the third inequality in (4.5.16).

The analyticity and boundedness of $\mathcal{R}$ on $\mathcal{D}$ follows from Lemma 4.5.1.

In order to bound the inverse of $\mathcal{L}$, let X be a vector field in $\mathbb{P}\mathcal{A}_{\rho}$. Then X can be written as $X(x, y) = (u, My + v)$, and

$$(\mathcal{L}^{-1}X)(x, y) = \eta(Tu, \bar{T}M\bar{T}^{-1}y + \mu\bar{T}v)\,. \tag{4.5.22}$$

This implies the last inequality in (4.5.16). **QED**

Due to the potentially large factor μ^{-1} in the third inequality of (4.5.16), we will choose the domain of $\mathcal{R}$ to be of the form

$$\|\mathbb{P}(X-K)\|_\varrho < r\,, \qquad \|(\mathbb{I}-\mathbb{P})X\|_\varrho < r\,, \qquad \|(\mathbb{I}-\mathbb{E})X\|_\varrho < r\delta\,, \tag{4.5.23}$$

with $0 < r \leq R(\sigma/\gamma)^2$, and with small $\delta > 0$ which will be determined later.

Definition 4.5.6 *Given $\gamma \geq 1$, we call $(\mu,\sigma,\tau,r,\delta)$ proper RG parameters if $r \leq R(\sigma/\gamma)^2$, and if μ,σ,τ are positive numbers that satisfy $\mu,\sigma,\tau < 1$ and the conditions* (4.5.2)*. We say that the pair (T,ω) is compatible with these parameters if the conditions* (4.5.1) *and* (4.3.1) *are satisfied as well. The open subset $\mathcal{D}$ of $\mathcal{A}_\varrho$ defined by equation* (4.5.23) *will be referred to as the domain of $\mathcal{R}$.*

4.6 Infinitely renormalizable vector fields

Our goal here is to compose RG transformations of the type described above. A sequence of scaling maps is constructed using a multidimensional generalization of the continued fraction algorithm, that we describe first.

4.6.1 A multidimensional continued fraction algorithm

We give a brief description of a multidimensional continued fractions expansion of Khanin, Lopes-Dias and Marklof [41], which is based on the work of Lagarias [53] and Kleinbock and Margulis [43] on geodesic flows on homogeneous spaces.

Let F be a fundamental domain for the left action of $\Gamma = \mathrm{SL}(d,\mathbb{Z})$ on $G =$

$\mathrm{SL}(d, \mathbb{R})$. Consider the one-parameter subgroup of G, generated by the matrices

$$E^t = \mathrm{diag}\big(e^{-t}, \ldots, e^{-t}, e^{(d-1)t}\big), \qquad t \in \mathbb{R}, \tag{4.6.1}$$

and the corresponding flow on the quotient space $\Gamma\backslash G$, defined by $\Gamma W \mapsto \Gamma W E^t$. Given a Diophantine vector $\omega \in \mathbb{R}^d$, define $W \in G$ to be the matrix obtained from the $d \times d$ identity matrix by replacing its last column vector by a constant multiple of ω whose last component is 1. Then, for every $t \in \mathbb{R}$, there exists a unique matrix $P(t) \in \Gamma$ such that $P(t)WE^t$ belongs to F. To a given sequence of "stopping times" $0 < t_1 < t_2 < \ldots$, we can now associate a sequence of matrices $P_n = P(t_n)$. The corresponding matrices T_n and vectors ω_n are defined as follows.

Given a unit vector $\omega \in \mathbb{R}^d$, we define $\omega_0 = \omega$, $\lambda_0 = 1$, and

$$T_n = P_{n-1}P_n^{-1}, \qquad \lambda_n = \|P_n\omega_0\|, \qquad \omega_n = \lambda_n^{-1}P_n\omega_0, \tag{4.6.2}$$

for all $n \in \mathbb{N}$.

The following estimates from [41], will be used later on. Let $t_0 = 0$, and define $t_n' = t_n - t_{n-1}$ for all positive integers n. Let $\theta = \beta/(d+\beta)$.

We assume that ω is Diophantine in the sense of Definition 4.1.1.

Theorem 4.6.1 (Khanin, Lopes-Dias and Marklof [41]) *There exists a constant $c_0 > 0$, depending only on the Diophantine constants β and C, such that for all $n > 0$, and for all vectors $\xi \in \mathbb{R}^d$ that are perpendicular to ω_{n-1},*

$$\begin{aligned} \|T_n\| &\le c_0 \exp\{(d-1)(1-\theta)t_n' + d\,\theta\,t_n\}, \\ \|T_n^{-1}\| &\le c_0 \exp\{(1-\theta)t_n' + d\,\theta\,t_n\}, \\ |T_n^*\xi| &\le c_0 \exp\{-(1-\theta)t_n' + d\,\theta t_{n-1}\}|\xi|. \end{aligned} \tag{4.6.3}$$

4.6.2 Sequence of renormalization transformations

In the following we assume that the pairs (T_n, ω_{n-1}), for $n \in \mathbb{N}$, have been obtained from the multidimensional continued fraction algorithm described above. We also assume that each pair is compatible with some proper set of RG parameters $(\mu_n, \sigma_n, \tau_n, r_{n-1}, \delta_{n-1})$. Thus, we can define the corresponding RG transformation $\mathcal{R}_n : \mathcal{D}_{n-1} \to \mathcal{A}_\varrho$. Notice that the normalization constant η_n for $\mathcal{R}_n$ is given by $\eta_n = \lambda_n / \lambda_{n-1}$.

In addition, let us define $\widetilde{\mathcal{R}}_n = \mathcal{R}_n \circ \mathcal{R}_{n-1} \circ \ldots \circ \mathcal{R}_1$. The domain $\widetilde{\mathcal{D}}_{n-1}$ of the combined RG transformation $\widetilde{\mathcal{R}}_n$ is defined inductively as the set of all vector fields in the domain of $\widetilde{\mathcal{R}}_{n-1}$ that are mapped under $\widetilde{\mathcal{R}}_{n-1}$ into the domain of $\mathcal{D}_{n-1}$ of $\mathcal{R}_n$. By Theorem 4.5.5, these domains are open and non-empty, and the transformations $\widetilde{\mathcal{R}}_n$ are analytic.

The following theorem shows that the renormalization parameters can be chosen such that there exist open sets of infinitely renormalizable vector fields.

Theorem 4.6.2 *Let $\alpha > \max\{3/2, \beta\}$, $m > 2\alpha + 7$, and $\gamma \geq 2\alpha + 2$ be given. There exist real numbers $b, C > 0$, a decreasing sequence of proper RG parameters $(\mu_n, \sigma_n, \tau_n, r_{n-1}, \delta_{n-1})$ satisfying for all $n \in \mathbb{N}$,*

$$\sigma_2 = \sigma_1^\alpha, \qquad \sigma_{n+2} = \sigma_{n+1}^{1+\alpha}, \qquad \mu_n = \sigma_n^m, \qquad r_n = \tfrac{1}{5} r_{n-1} \sigma_{n+1}^2, \tag{4.6.4}$$

and for every every Diophantine vector $\omega \in \mathbb{R}^d$ a sequence of matrices $P_n \in \mathrm{SL}(d, \mathbb{Z})$ yielding pairs (T_n, ω_{n-1}) that are compatible with the RG parameters, and an open neighborhood B of $K = (\omega, 0)$ in $\mathcal{A}_\varrho$ such that the following holds. B contains a ball of radius b, centered at K. The set $\mathcal{W} = B \cap_n \widetilde{\mathcal{D}}_n$ of infinitely renormalizable vector fields is the graph of an analytic function $W : (\mathbb{I} - \mathbb{P})B \to$

$\mathbb{P}B$*, satisfying* $W(0) = K$ *and* $DW(0) = 0$*. For each* $X \in \mathcal{W}$ *and* $n \geq 1$*,*

$$
\begin{aligned}
\left\|\widetilde{\mathcal{R}}_n(X) - K_n\right\|_\varrho &\leq C\sigma_n^{m-7-2\alpha} r_n \|(\mathbb{I}-\mathbb{P})X\|_\varrho \,, \\
\left\|\mathbb{P}[\widetilde{\mathcal{R}}_n(X) - K_n]\right\|_\varrho &\leq C\sigma_n^{2(m-7-2\alpha)} r_n \|(\mathbb{I}-\mathbb{P})X\|_\varrho^2 \,, \qquad (4.6.5) \\
\left\|(\mathbb{I}-\mathbb{E})\widetilde{\mathcal{R}}_n(X)\right\|_\varrho &\leq C\sigma_n^{(m-1)\gamma-6-2\alpha} r_n \|(\mathbb{I}-\mathbb{E})X\|_\varrho \,.
\end{aligned}
$$

Proof: Choose

$$
t_n = c(1+\alpha)^n \,, \qquad n = 1, 2, \ldots \,, \qquad (4.6.6)
$$

with $\alpha > \beta$ fixed, and with $c > 0$ to be determined. Define $c_1 = 2c_0\sqrt{d}$ and

$$
\sigma_n = \exp\{-dt_n'\} \,, \qquad \tau_n = c_1 \exp\{-(1-\theta)t_n' + d\,\theta t_{n-1}\} \,. \qquad (4.6.7)
$$

Then Theorem 4.6.1 guarantees that the conditions (4.3.1) and (4.5.1) are satisfied. By using that $t_1' = t_1$ and $t_n' = \frac{\alpha}{1+\alpha} t_n$ for $n > 1$, we obtain the bounds

$$
\sigma_n \leq \exp\{-d\tfrac{\alpha}{1+\alpha} t_n\} \,, \qquad \tau_n \leq c_1 \exp\{-\epsilon t_n\} \,, \qquad (4.6.8)
$$

with $\epsilon = \frac{1-\theta}{1+\alpha}(\alpha - \beta) > 0$. Let now $\mu_n = \sigma_n^m$ with $m > 1$ fixed. Then it is clear that the conditions (4.5.2) are satisfied as well, for any $\gamma > 0$, provided that c is chosen sufficiently large. Here and in what follows any condition that is said to hold for large values of c, is implicitly being satisfied by choosing c as large as necessary.

Now, let $r_0 = R(\sigma_1/\gamma)^2$, and define r_n for $n \in \mathbb{N}$ as in (4.6.4). It is easy to show by induction that $r_n \leq R(\sigma_{n+1}/\gamma)^2$, for all $n \in \mathbb{N}_0$. Thus, we have shown that $(\mu_n, \sigma_n, \tau_n, r_{n-1}, \delta_{n-1})$ are proper RG parameters, in the sense of Definition 4.5.6, and that (T_n, ω_{n-1}) is compatible with these parameters. This is independent of the choice of $\delta_{n-1} > 0$, which we will describe below.

Consider now the rescaled RG transformations R_n , defined by the equation

$$R_n(Z) = r_n^{-1}\big[\mathcal{R}_n(K_{n-1} + r_{n-1}Z) - K_n\big]\,. \tag{4.6.9}$$

The domain of R_n is given by (4.5.23), with $r = 1$ and $\delta = \delta_{n-1}$, and with K replaced by the zero vector field. The restriction of R_n to $\mathbb{P}\mathcal{A}_\rho$, which is linear, will be denoted by L_n . By using the bound on $\mathcal{L}_n^{-1}$ from Theorem 4.5.5, we obtain

$$\|L_n^{-1}\| \le 1/5\,, \tag{4.6.10}$$

if $\alpha > 3/2$. Here, we have used also that $\sigma_n < \|T_n\|^{-1} \le \eta_n \le \|T_n^{-1}\| < \sigma_n^{-1}$, and that $C_0\tau_n \le 1$, if c is chosen sufficiently large. The same inequalities, together with Theorem 4.5.5, also imply that

$$\begin{aligned}
\|(\mathbb{I}-\mathbb{E})R_n(Z)\|_\varrho &\le \varepsilon_n\|(\mathbb{I}-\mathbb{E})Z\|_\varrho\,, \quad \varepsilon_n = 5C_0\gamma\sigma_n^{(m-1)\gamma-6-2\alpha}\,,\\
\|(\mathbb{I}-\mathbb{P})R_n(Z)\|_\varrho &\le \vartheta_n\|(\mathbb{I}-\mathbb{P})Z\|_\varrho\,, \quad \vartheta_n = 5C_0\gamma\sigma_n^{m-7-2\alpha}\,,\\
\|\mathbb{P}R_n(Z) - R_n(\mathbb{P}Z)\|_\varrho &\le \varphi_n\delta_{n-1}\|(\mathbb{I}-\mathbb{E})Z\|_\varrho\,, \quad \varphi_n = 5C_0\gamma^3\sigma_n^{-m-7-2\alpha}\,,
\end{aligned} \tag{4.6.11}$$

for all Z in the domain of R_n . Assume now that $m > 7 + 2\alpha$ and $\gamma \ge 1$. Then $\varepsilon_n \le \vartheta_n \le 1/5$, if c is sufficiently large. Furthermore, by setting for $n \in \mathbb{N}$,

$$\delta_{n-1} = (25C_0\gamma^3)^{-1}\sigma_n^{m+7+2\alpha}\,, \tag{4.6.12}$$

we obtain $\varphi_n\delta_{n-1} \le 1/5$. It is easy to check that $\varepsilon_n\delta_{n-1} \le \delta_n$, if $\gamma \ge 2\alpha + 2$, provided again that c has been chosen sufficiently large.

At this point we have verified the hypotheses of Theorem 4.7.5, with $\varepsilon = \vartheta = 1/5$. This includes the condition (4.7.17), since the third inequality in (4.6.11) remains true if δ_{n-1} is replaced by $\|(\mathbb{I}-\mathbb{E})Z\|_{\varrho}$. The assertions now follow from Theorem 4.7.5.

We note that the second bound in (4.6.5) is not strictly needed to make the analysis work. The first bound, with a larger value of m could be used instead. **QED**

The above also proves Theorem 4.1.1, except for the statement concerning the restriction of W to a particular class of vector fields. The fact that W preserves the class of a vector field (Hamiltonian, divergence-free, symmetric, or reversible) follows directly from the fact that the renormalization operator $\mathcal{R}$ is class-preserving. The elimination step $X \mapsto \mathcal{U}_X^* X$ is class-preserving by construction as can be seen from the discussion at the end of Section 4.4. The scaling $X \mapsto \mathcal{T}^*\mathcal{S}_\mu^* X$ is class-preserving as well.

4.7 A stable manifold theorem

In this section, we state and prove a "stable manifold" theorem for sequences of maps of the type encountered in renormalization.

4.7.1 Assumptions on a sequence of maps $\{R_n\}$

For every $n \in \mathbb{N}_0$, let $\mathcal{X}_n$ be a complex Banach space and let $\mathbb{E}_n$ and $\mathbb{P}_n$ be continuous linear projections on $\mathcal{X}_n$, satisfying $\mathbb{P}_n\mathbb{E}_n = \mathbb{E}_n\mathbb{P}_n = \mathbb{P}_n$ and $\|\mathbb{E}_n\| = \|\mathbb{I}-\mathbb{E}_n\| = 1$. For every $n \in \mathbb{N}$, let R_n be a bounded analytic map, from an open neighborhood D_{n-1} of the origin in $\mathcal{X}_{n-1}$ into $\mathcal{X}_n$, with the following properties: $R_n\mathbb{P}_{n-1}$ is linear, and the restriction L_n of this linear operator to $\mathbb{P}_{n-1}\mathcal{X}_{n-1}$ is invertible. Furthermore, we assume that there exist positive numbers $\vartheta_n \le \vartheta < 1$ and $\varepsilon_n \le \varepsilon = (1-\vartheta)/4$, such that for all $x \in D_{n-1}$,

$$\begin{aligned}
\|(\mathbb{I}-\mathbb{E}_n)R_n(x)\| &\leq \varepsilon_n\|(\mathbb{I}-\mathbb{E}_{n-1})x\|\,, \\
\|(\mathbb{I}-\mathbb{P}_n)R_n(x)\| &\leq \vartheta_n\|(\mathbb{I}-\mathbb{P}_{n-1})x\|\,, \\
\|\mathbb{P}_n R_n(x) - L_n\mathbb{P}_{n-1}x\| &\leq \varepsilon\|(\mathbb{I}-\mathbb{E}_{n-1})x\|\,, \qquad (4.7.1)\\
\|L_n^{-1}\| &\leq \vartheta\,.
\end{aligned}$$

Consider now the composed maps $\widetilde{R}_n = R_n \circ R_{n-1} \circ \ldots \circ R_1$. The domain of $\widetilde{R}_1$ is taken to be $\widetilde{D}_0 = D_0$, and for $n \in \mathbb{N}$, the domain $\widetilde{D}_n$ of $\widetilde{R}_{n+1}$ is defined inductively as the subset of $\widetilde{D}_{n-1}$ that is mapped into D_n by $\widetilde{R}_n$.

We will assume that the domain D_{n-1} of R_n is given by conditions

$$\|\mathbb{P}_{n-1}x\| < 1\,, \qquad \|(\mathbb{I}-\mathbb{P}_{n-1})x\| < 1\,, \qquad \|(\mathbb{I}-\mathbb{E}_{n-1})x\| < \delta_{n-1}\,, \qquad (4.7.2)$$

where $\{\delta_n\}$, is a sequence of positive numbers that satisfy $\delta_n \geq \varepsilon_n\delta_{n-1}$, for $n \in \mathbb{N}$.

Notice that, by our assumptions (4.7.1), if x belongs to the domain of R_n, and if $\mathbb{P}_n R_n(x)$ has norm less than one, then $R_n(x)$ belongs to the domain of R_{n+1}. This shows that, for $n \in \mathbb{N}$,

$$\widetilde{D}_n = \{x \in \widetilde{D}_{n-1} : \|\mathbb{P}_n\widetilde{R}_n(x)\| < 1\}\,. \qquad (4.7.3)$$

4.7.2 Stable manifold for the sequence $\{R_n\}$

Let $S_n = \mathbb{P}_n\mathcal{X}_n$, and denote by b_n the open unit ball in S_n, centered at the origin. Define $\mathcal{F}_n$ to be the space of analytic functions $f : b_n \to \mathcal{X}_n$, equipped with the sup-norm $\|f\| = \sup_{s\in b_n}\|f(s)\|$. Denote by I_n the inclusion map of b_n into $\mathcal{X}_n$.

Notice that, if $f \in \mathcal{F}_{n-1}$ satisfies

$$\mathbb{P}_{n-1} f = I_{n-1}\,, \qquad \|f - I_{n-1}\| < 1\,, \qquad \|(\mathbb{I} - \mathbb{E}_{n-1}) \circ f\| < \delta_{n-1}\,, \tag{4.7.4}$$

then $f(s)$ belongs to the domain of R_n, for all $s \in b_{n-1}$. For such functions f, define

$$Y_{n,f} = \mathbb{P}_n(R_n \circ f)\,. \tag{4.7.5}$$

Proposition 4.7.1 *Assume that $f \in \mathcal{F}_{n-1}$ satisfies* (4.7.4)*. Then $Y_{n,f} : b_{n-1} \to S_n$ has a unique right inverse $Y_{n,f}^{-1} : b_n \to b_{n-1}$. Both $Y_{n,f}$ and its inverse depend analytically on f, on the domain defined by* (4.7.4)*. Furthermore,*

$$\begin{aligned} \|Y_{n,f} - \vartheta_n\| &\leq \varepsilon \|(\mathbb{I} - \mathbb{E}_{n-1}) \circ f\|\,, \\ \|Y_{n,f}^{-1} - L_n^{-1}\| &\leq \vartheta\varepsilon \|(\mathbb{I} - \mathbb{E}_{n-1}) \circ f\|\,. \end{aligned} \tag{4.7.6}$$

Proof: Let $U = Y_{n,f} - L_n$. By the third condition in (4.7.1) we have

$$\|U(s)\| = \|\mathbb{P}_n R_n(f(s)) - L_n \mathbb{P}_{n-1} f(s)\| \leq \varepsilon \|(\mathbb{I} - \mathbb{E}_{n-1}) f(s)\|\,, \tag{4.7.7}$$

for all $s \in b_{n-1}$. This implies the first bound in (4.7.6).

By our assumption on f and ε, we have $\|U\| \leq \varepsilon \leq r/2$, where $r = (1-\vartheta)/2$. If $s \in S_{n-1}$ is of norm smaller than or equal to ϑ and $h \in S$ is of norm one, then by Cauchy's formula

$$\|DU(s)h\| \leq r^{-1} \sup_{|z|=r} \|U(s + zh)\| \leq r^{-1} \|U\| \leq 1/2\,. \tag{4.7.8}$$

The equation for the right inverse $L_n^{-1} + V$ of $L_n + U$ can be written as $\psi(V) = V$, with ψ defined by $\psi(V) = -L_n^{-1} U \circ (L_n^{-1} + V)$. Consider the space of analytic functions $V : b_n \to S_{n-1}$, equipped with the sup-norm. Denote by B the closed ball of radius r in this space, centered at the origin. Then ψ is analytic on B, with derivative given by

$$D\psi(V)h = -L_n^{-1}((DU) \circ (L_n^{-1} + V))h\,. \tag{4.7.9}$$

From the inequalities (4.7.8), we see that $\|D\psi(V)\| < 1/2$, for all $V \in B$. Since $\|\psi(0)\| \leq r/2$, the map ψ is a contraction on B, and thus has a (unique) fixed point in B. This fixed point V satisfies $\|V\| = \|\psi(V)\| \leq \|L_n^{-1}U\|$, which implies the second inequality in (4.7.6). The analyticity of $U \mapsto V$ follows from the uniform convergence of $\psi^n(0) \to V$ for $\|U\| \leq r/2$. **QED**

This proposition allows us to define the maps

$$\mathfrak{R}_n(f) = R_n \circ f \circ Y_{n,f}^{-1}\,, \qquad \widetilde{\mathfrak{R}}_n = \mathfrak{R}_n \circ \mathfrak{R}_{n-1} \circ \ldots \circ \mathfrak{R}_1\,. \tag{4.7.10}$$

Notice that $\mathbb{P}_n\mathfrak{R}_n(f) = I_n$. In particular, since $R_n \circ \mathbb{P}_{n-1} = \mathbb{P}_n \circ R_n \circ \mathbb{P}_{n-1}$, by the second condition in (4.7.1), we have $\mathfrak{R}_n(I_{n-1}) = I_n$. The domain of $\mathfrak{R}_n$ is the set of all $f \in \mathcal{F}_{n-1}$ satisfying (4.7.4).

For $n \in \mathbb{N}$, define

$$\vartheta^{(n)} = \prod_{j=1}^{n} \vartheta_j\,, \qquad \varepsilon^{(n)} = \prod_{j=1}^{n} \varepsilon_j\,. \tag{4.7.11}$$

Lemma 4.7.2 *If f_0 belongs to the domain of $\mathfrak{R}_1$, then $\widetilde{\mathfrak{R}}_n(f_0)$ is well-defined for all $n \in \mathbb{N}$, and*

$$\|\widetilde{\mathfrak{R}}_n(f_0) - I_n\| \leq \vartheta^{(n)}\|f_0 - I_0\|\,. \tag{4.7.12}$$

Proof: Given $n \in \mathbb{N}$, let f be an arbitrary function in the domain of $\mathfrak{R}_n$, and let $f' = \mathfrak{R}_n(f)$. Consider a fixed, but arbitrary, $s \in b_n$ and define $s' = Y_{n,f}^{-1}(s)$. By Proposition 4.7.1, s' belongs to b_{n-1}. Thus, the second condition in (4.7.1) implies

$$\begin{aligned} \|f'(s) - s\| &= \|(\mathbb{I} - \mathbb{P}_n) R_n(f(s'))\| \\ &\leq \vartheta_n \|(\mathbb{I} - \mathbb{P}_{n-1}) f(s')\| = \vartheta_n \|f(s') - s'\| \,. \end{aligned} \tag{4.7.13}$$

This shows that $\|f' - I_n\| \leq \vartheta_n \|f - I_{n-1}\|$. In addition, we have $\mathbb{P}_n f' = I_n$, by the definition of $\mathfrak{R}_n$, and $\|(\mathbb{I} - \mathbb{E}_n) \circ f'\| \leq \varepsilon_n \delta_{n-1}$, by the first inequality in (4.7.1). Thus, since $\vartheta_n < 1$ and $\varepsilon_n \delta_{n-1} \leq \delta_n$, the function f' belongs to the domain of $\mathfrak{R}_{n+1}$. This proves the claim. **QED**

This lemma shows that the domain of $\widetilde{\mathfrak{R}}_n$ can be taken to be the domain of $\mathfrak{R}_1$. If f_0 is any function in this domain, define for $n > m \geq 0$,

$$f_n = \widetilde{\mathfrak{R}}_n(f_0)\,, \qquad Y_n = Y_{n,f_{n-1}}\,, \qquad Z_{m,n} = Y_{m+1}^{-1} \circ \ldots \circ Y_{n-1}^{-1} \circ Y_n^{-1}\,. \tag{4.7.14}$$

Proposition 4.7.3 *For every f in the domain of $\mathfrak{R}_1$, there exists a unique sequence $m \mapsto z_m \in b_m$, satisfying*

$$z_{m-1} = Y_m^{-1}(z_m)\,, \tag{4.7.15}$$

for $m \in \mathbb{N}$, and this sequence is given by the limits $z_m = \lim_{n\to\infty} Z_{m,n}(0)$. The maps $f \mapsto z_m$ are analytic on the domain of $\mathfrak{R}_1$.

Proof: First, we note that it suffices to prove the claims for $m \geq N$, where N is a fixed but arbitrary positive integer.

Let $f_0 = f$. Since $\|Y_n^{-1} - L_n^{-1}\| \le \vartheta^n$ by Proposition 4.7.1 and Lemma 4.7.2, there exist an integer $N > 0$, and two positive real numbers $r, r' < 1$, both independent of f_0, such that Y_n^{-1} maps b_n into to $r' b_{n-1}$ and contracts distances by a factor smaller than or equal to r, whenever $n \ge N$.

In the remaining part of the proof, we assume that $N \le m < n$. Consider, an arbitrary sequence $n \mapsto s_n \in b_n$, with the property that s_n belongs to the closure of $r' b_n$. Notice that if a sequence $n \mapsto z_n \in b_n$ satisfies (4.7.15), then it automatically has this property. Define $s_{m,n} = Z_{m,n}(s_n)$. Then $\|s_{m,k} - s_{m,n}\| < 2r^{n-m}$, whenever $k > n$. This shows that $n \mapsto s_{m,n}$ converges as $n \to \infty$, and that the limit $\tilde{s}_m$ is independent of the sequence $\{s_n\}$. In particular, we see that $\tilde{s}_m = z_m$ by choosing $s_n = 0$, for all n. The identities (4.7.15) are obtained by choosing $s_n = z_n$ for all n.

By Proposition 4.7.1, the maps $f \mapsto s_{m,n} = Z_{m,n}(0)$ are analytic on the domain of $\mathfrak{R}_1$. The analyticity of $f \mapsto z_m$ now follows from the uniform convergence of $s_{m,n} \to z_m$. **QED**

Corollary 4.7.4 *Let f be a family in the domain of $\mathfrak{R}_1$, and let $s \in b_0$. Then $f(s)$ belongs to $\mathcal{W}_0 := \bigcap_{n=0}^{\infty} \widetilde{D}_n$, if and only if $s = z_0(f)$.*

Proof: Consider first $x = f(z_0)$. Then $x \in D_0$, since f belongs to the domain of $\mathfrak{R}_1$. Let $x_n = f_n(z_n)$, for all $n \in \mathbb{N}$. By the definition of $\mathfrak{R}_n$, and by Proposition 4.7.3, we have $x_n = R_n(x_{n-1}) = \widetilde{R}_n(x)$. Furthermore, $\mathbb{P}_n x_n = \mathbb{P}_n f_n(z_n) = z_n \in b_n$, and thus x belongs to the set $\widetilde{D}_n$ described in (4.7.3). This shows that $x \in \mathcal{W}_0$.

Consider now a fixed $s = s_0 \in b_0$, and assume that $x_0 = f(s_0)$ belongs to $\mathcal{W}_0$. We define $x_n = \widetilde{\mathfrak{R}}_n(x)$, for all $n \in \mathbb{N}$. Then, $s_n = \mathbb{P}_n x_n$ belongs to b_n. Set $f_0 = f$. Proceeding by induction, let $n > 0$, and assume that $x_{n-1} = f_{n-1}(s_{n-1})$. Since $s_n = Y_n(s_{n-1})$, and since Y_n has a unique right inverse on b_n, by Proposition 4.7.1, we have $s_{n-1} = Y_n^{-1}(s_n)$. As a result, $x_n = f_n(s_n)$. This shows that $s_{n-1} = Y_n^{-1}(s_n)$ holds for all $n \in \mathbb{N}$, and thus $s_n = z_n$, by Proposition 4.7.3. **QED**

Theorem 4.7.5 (Stable manifold) *Let $\{R_n\}$, $n \in \mathbb{N}$, be a sequence of maps which satisfies assumptions of Section 4.7.1. Then, $\mathcal{W}_0 = \bigcap_{n=0}^{\infty} \widetilde{D}_n$ is the graph of an analytic function $W_0 : (\mathbb{I} - \mathbb{P}_0)D_0 \to \mathbb{P}_0 D_0$, satisfying $W_0(0) = 0$, and*

$$\begin{aligned} \|\widetilde{R}_m(x)\| &\leq \left[\vartheta^{(m)} + \varepsilon^{(m)}\right] \|(\mathbb{I} - \mathbb{P}_0)x\|\,, \\ \|(\mathbb{I} - \mathbb{E}_m)\widetilde{R}_m(x)\| &\leq \varepsilon^{(m)} \|(\mathbb{I} - \mathbb{E}_0)x\|\,, \end{aligned} \tag{4.7.16}$$

for all $x \in \mathcal{W}_0$. Furthermore, if the third condition in (4.7.1) *is replaced by a stronger one,*

$$\|\mathbb{P}_n R_n(x) - L_n \mathbb{P}_{n-1} x\| \leq \varphi_n \|(\mathbb{I} - \mathbb{E}_{n-1})x\|^2\,, \tag{4.7.17}$$

with $\varphi_n \delta_{n-1} \leq \varepsilon$, then $DW_0(0) = 0$, and

$$\|\mathbb{P}_m \widetilde{R}_m(x)\| \leq \left[\vartheta^{(m)}\right]^2 \|(\mathbb{I} - \mathbb{P}_0)x\|^2\,. \tag{4.7.18}$$

Recall that $\vartheta^{(m)}$ and $\varepsilon^{(m)}$ are defined as in (4.7.11).

Proof: Denote by B_0' the unit ball in $(\mathbb{I} - \mathbb{P}_0)\mathcal{X}_0$, centered at the origin. To a point $x \in B_0'$, we associate the family $f : s \mapsto s + x$. This family belongs to the domain of $\mathfrak{R}_1$. Now define $W_0(x) = z_0(f)$. By Corollary 4.7.4, $x + s = f(s)$ belongs to $\mathcal{W}_0$ if and only if $s = W_0(x)$. This shows that $\mathcal{W}_0$ is the graph of W_0 over B_0'. The analyticity of W_0 follows from the analyticity of z_0. Furthermore, we have $W_0(0) = z_0(I_0) = 0$.

The second bound in (4.7.16) follows from the first condition in (4.7.1). In order to prove the first bound, consider the family $f_0(s) = s + (\mathbb{I} - \mathbb{P}_0)x$, the associated functions f_n and Y_n defined in (4.7.14), and the parameters z_n described in Proposition 4.7.3. Then, $\widetilde{R}_n(x) = f_n(z_n)$, for all $n \in \mathbb{N}_0$. By Lemma 4.7.2, we have

$$\|f_m(z_m) - z_m\| \leq \vartheta^{(m)} \|(\mathbb{I} - \mathbb{P}_0)x\|, \tag{4.7.19}$$

and, by Proposition 4.7.1 and the second inequality in (4.7.16),

$$\begin{aligned}
\|z_m - L_{m+1}^{-1} \cdots L_n^{-1} z_n\| &= \left\| \sum_{k=m}^{n-1} L_{m+1}^{-1} \cdots L_k^{-1} [Y_{k+1}^{-1} - L_{k+1}^{-1}](z_{k+1}) \right\| \\
&\leq \sum_{k=m}^{n-1} \vartheta^{k-m} \vartheta \varepsilon\, \varepsilon^{(k)} \|(\mathbb{I} - \mathbb{E}_0)x\| \qquad (4.7.20) \\
&\leq \frac{\vartheta \varepsilon}{1 - \vartheta \varepsilon} \varepsilon^{(m)} \|(\mathbb{I} - \mathbb{E}_0)x\|,
\end{aligned}$$

whenever $0 \leq m < n$. These two inequalities, together with the fact that $L_{m+1}^{-1} \cdots L_n^{-1} z_n$ tends to zero as $n \to \infty$, imply the first bound in (4.7.16).

Next, assume that (4.7.17) holds. Then the equation (4.7.7) shows that for each $n > 0$, the map $f \mapsto Y_{n,f}$ has a vanishing derivative at $f = \mathrm{I}_{n-1}$. By the definition of W_0, this implies that $DW_0(0) = 0$.

Let now $x_0 \in \mathcal{W}_0$. Then $x_0 = u + W_0(u)$ with $u = (\mathbb{I} - \mathbb{P}_0)x_0$. Assume that $u \neq 0$. Let ℓ be a continuous linear functional on $\mathcal{X}_0$ of norm one, such that $\ell(W_0(u)) = \|W_0(u)\|$. Define $g(z) = \ell(W_0(zu/\|u\|))$ for all z in the complex unit disk $|z| < 1$. Since W_0 and DW_0 vanish at the origin, $z \mapsto z^{-2}g(z)$ defines an analytic function on the unit disk, and by Schwarz's lemma, this function is bounded in modulus by 1. Here, we have used that W_0 has norm less than one on its domain. This shows that $\|W_0(u)\| = g(\|u\|) \leq \|u\|^2$, or in other words, that $\|\mathbb{P}_0 x_0\| \leq \|(\mathbb{I} - \mathbb{P}_0)x_0\|^2$.

Finally, let $m \in \mathbb{N}$ and consider the stable manifold $\mathcal{W}_m$ for the shifted sequence of maps $R_m, R_{m+1}, \ldots$. Clearly, $x_m = \tilde{R}_m(x_0)$ belongs to $\mathcal{W}_m$. The same arguments as above show that $\|\mathbb{P}_m x_m\| \leq \|(\mathbb{I} - \mathbb{P}_m)x_m\|^2$. The bound (4.7.18) now follows from the second condition in (4.7.1). **QED**

4.8 Construction of invariant tori

In this section, we apply the previously constructed renormalization scheme for vector fields to a KAM-type problem, i.e. the construction of invariant tori for near-integrable vector fields. Following the ideas for the construction of invariant tori for Hamiltonians from the previous chapter, the construction of invariant tori for vector fields is based on the relation between an invariant torus of a vector field X and the corresponding torus of the renormalized vector field $\mathcal{R}(X)$.

4.8.1 Preliminaries

We start with an informal discussion of this relation. Let $X \in \mathcal{A}_\varrho$. Notice that $\mathcal{R}(X)$ is obtained from X by a change of coordinates (that depends on X), combined with a rescaling of time. Thus, the flow for $\mathcal{R}(X)$ is related to the flow for X by

$$\Lambda_X \circ \Phi^t_{\mathcal{R}(X)} = \Phi^{\eta^{-1}t}_X \circ \Lambda_X\,, \qquad \Lambda_X = \mathcal{U}_X \circ \mathcal{S}_\mu \circ \mathcal{T}\,. \tag{4.8.1}$$

In particular, $\mathcal{T} \circ \Phi^t_{\mathcal{R}(K)} = \Phi^{\eta^{-1}t}_K \circ \mathcal{T}$ on D_0. The identity (4.8.1) can also be used to relate an invariant torus for X to an invariant torus for $\mathcal{R}(X)$. To this end, if F is any map from D_0 into the domain of Λ_X, define

$$\mathcal{M}_X(F) = \Lambda_X \circ F \circ \mathcal{T}^{-1}\,. \tag{4.8.2}$$

Assume that $\mathcal{R}(X)$ has an invariant torus $\widetilde{\Gamma}$ with frequency vector $\widetilde{\omega} = \eta^{-1}T^{-1}\omega$, taking values in the domain of Λ_X, and define $\Gamma = \mathcal{M}_X(\widetilde{\Gamma})$. Then, we obtain

$$\begin{aligned} \Gamma \circ \Phi_K^t &= \Lambda_X \circ \widetilde{\Gamma} \circ \mathcal{T}^{-1} \circ \Phi_K^t = \Lambda_X \circ \widetilde{\Gamma} \circ \Phi_{\mathcal{R}(K)}^{\eta t} \circ \mathcal{T}^{-1} \\ &= \Lambda_X \circ \Phi_{\mathcal{R}(X)}^{\eta t} \circ \widetilde{\Gamma} \circ \mathcal{T}^{-1} = \Phi_X^t \circ \Lambda_X \circ \widetilde{\Gamma} \circ \mathcal{T}^{-1} = \Phi_X^t \circ \Gamma\,, \end{aligned} \tag{4.8.3}$$

by using (4.8.1), together with the fact that $\mathcal{R}(K) = (\widetilde{\omega}, 0)$. This shows that Γ is an invariant torus for X with frequency vector ω.

In order to make these identities more precise, we need to estimate the difference $Y(t) = \Phi_X^t - \Phi_K^t$ between the flow for a vector field X and the flow for $K = (\omega, 0)$. This can be done by solving the integral equation

$$Y(t) = \int_0^t [(X - K) \circ \Phi_K^s] \circ [\mathrm{I} + Y(s)]\, ds\,. \tag{4.8.4}$$

Notice that $\Phi_K^t = \mathrm{I} + tK$ and $\mathrm{I} + Y(t) = \Phi_K^{-t} \circ \Phi_X^t$.

Proposition 4.8.1 *Let τ be a positive real number and X a vector field in $\mathcal{A}_\rho$, such that $\tau\|X - K\|_\rho < r < \rho$. Then the equation (4.8.4) has a unique continuous solution $t \mapsto Y(t) \in \mathcal{A}_{\rho-r}$ on the interval $|t| \le \tau$, and*

$$\|\Phi_X^t - \Phi_K^t\|_{\rho-r} \le \|t(X - K)\|_\rho\,. \tag{4.8.5}$$

Proof: By using Proposition 4.5.2 and the contraction mapping principle, the equation (4.8.4) is easily seen to have a unique continuous solution $t \mapsto Y(t) \in \mathcal{A}_{\rho-r}$ for t near 0. The solution can be continued as usual, as long as $\|Y(t)\|_{\rho-r} < r$. But on any interval containing zero, where $\|Y(t)\|_{\rho-r} < r$, we have by (4.8.4) the bound

$$\|Y(t)\|_{\rho-r} \le \|t(X - K)\|_\rho\,. \tag{4.8.6}$$

Here, we have used also that $Z \mapsto Z \circ \Phi_K^s$ is an isometry on $\mathcal{A}_\rho$. Thus, equation (4.8.4) has a continuous solution Y for all times t satisfying $\|t(X-K)\|_\rho < r$. **QED**

4.8.2 Existence of invariant tori

Consider now a fixed but arbitrary vector field X on the stable manifold $\mathcal{W}$ of our RG transformations $\{\mathcal{R}_n\}$. Let $X_0 = X$, and $X_n = \mathcal{R}_n(X_{n-1})$, for $n \in \mathbb{N}$. In order to simplify the notation, we will write $\mathcal{U}_k$ and $\mathcal{M}_{k+1}$ in place of $\mathcal{U}_{X_k}$ and $\mathcal{M}_{X_k}$, respectively. Our goal is to construct an appropriate sequence of functions $\Gamma_k : D_0 \to D_\varrho$, satisfying

$$\Gamma_{n-1} = \mathcal{M}_n(\Gamma_n) = \Lambda_n \circ \Gamma_n \circ \mathcal{T}_n^{-1}, \qquad \Lambda_n = \mathcal{U}_{n-1} \circ \mathcal{S}_{\mu_n} \circ \mathcal{T}_n, \tag{4.8.7}$$

for all positive integers n. Then, we will show that Γ_k is an invariant torus for X_k, with frequency vector ω_k, for each $k \geq 0$.

For every integer $n \geq 0$, define $\mathcal{B}_n$ to be the vector space $\mathcal{A}_0$, equipped with the norm

$$\|f\|'_n = ar_n^{-1}\|f\|_0 = ar_n^{-1}\sum_\nu \|f_\nu\|. \tag{4.8.8}$$

Here, a is some positive real number, to be specified later.

In the following, denote by B_n the unit ball in the affine space $\mathrm{I} + \mathcal{B}_n$, centered at the identity function I. Denote by $B_n/2$ the ball of radius $1/2$ centered at I in the same space.

In the following, assume that that α, m, and γ satisfy the conditions given in Theorem 4.6.2.

Proposition 4.8.2 *If a has been chosen sufficiently large, then there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, and a universal constant $C_1 > 0$, such* *that for every $X \in B \cap \mathcal{W}$, and for every $n \geq 1$, the map $\mathcal{M}_n$ is well-defined and analytic, as a function from B_n to B_{n-1}, and it takes values in $B_{n-1}/2$. Furthermore, $\|D\mathcal{M}_n(F)\| \leq C_1\sigma_n$, for all $F \in B_n$.*

Proof: Clearly, $\mathcal{M}_n$ is well-defined in some open neighborhood of I in $\mathcal{B}_n$, and

$$\mathcal{M}_n(F) = \mathrm{I} + g + (\mathcal{U}_{n-1} - \mathrm{I}) \circ (\mathrm{I} + g)\,, \qquad g = \mathcal{S}_{\mu_n} \circ \mathcal{T}_n \circ f \circ \mathcal{T}_n^{-1}\,, \tag{4.8.9}$$

where $f = F - \mathrm{I}$. By Theorem 4.4.5 and Theorem 4.6.2, for every integer $n \geq 2$, we have the bound

$$\begin{aligned} \|\mathcal{U}_{n-1} - \mathrm{I}\|_\rho &\leq C_2\sigma_n^{-1}\|\mathbb{I}^- X_{n-1}\|_\varrho \leq C_3\sigma_n^{-1}\sigma_{n-1}^{(m-1)\gamma-6-2\alpha} r_{n-1}\|(\mathbb{I} - \mathbb{E})X\|_\varrho \\ &\leq C_4\sigma_{n-1}^{(m-1)\gamma-7-3\alpha} r_{n-1}\|(\mathbb{I} - \mathbb{E})X\|_\varrho \leq C_5\sigma_n\, r_{n-1}\,, \end{aligned} \tag{4.8.10}$$

where C_2, C_3, C_4 and C_5 are some universal constants. The first inequality and the final bound in (4.8.10) are also valid for $n = 1$, if the neighborhood B of K is sufficiently small.

Recall that $\rho' < \rho < \varrho$ have been fixed. The composition with $\mathrm{I} + g$ in equation (4.8.9) is controlled by the bounds in Proposition 4.5.2, using that $\|g\|_0 \leq \sigma_n^{-1}a^{-1}r_n\|f\|_n' < \rho'$ independently of $n \in \mathbb{N}$, if a has been chosen sufficiently large. Here, and in what follows, we assume that $F \in B_n$.

By using that $r_n/r_{n-1} = \sigma_{n+1}^2/5$, we obtain $\|g\|_{n-1}' \leq \sigma_n/5$. When combined with (4.8.10), this yields the bound $\|\mathcal{M}_n(F) - \mathrm{I}\|_{n-1}' \leq \sigma_n/2$, if the constant C_5 has been chosen sufficiently small (depending on a). This condition can be satisfied if the neighborhood B of K is chosen sufficiently small.

When restricting $\mathcal{U}_{n-1}$ to the domain $D_{\rho'}$, we obtain a bound analogous to (4.8.10) for the derivative of $\mathcal{U}_{n-1}$. This shows that $\|D\mathcal{M}_n(F)\| \leq C_1\sigma_n$, for all

$F \in B_n$, and for all $n \geq 1$. Here C_1 is again a universal constant. This completes the proof. **QED**

Denote by Φ_n and Ψ_n the flows for the vector fields X_n and K_n , respectively.

Proposition 4.8.3 *Assume that $m > 7+2\alpha+p$ with $p > 0$. If a has been chosen sufficiently large, then there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, such that the following holds, for every $X \in B \cap \mathcal{W}$, and for every $n \geq 1$. If $F \in B_n/2$ and $|s| \leq \sigma_n^{-p}$, then $\Phi_n^s \circ F \circ \Psi_n^{-s}$ belongs to B_n .*

Proof: We will use the identity

$$\Phi_n^s \circ F \circ \Psi_n^{-s} = \mathrm{I} + f \circ \Psi_n^{-s} + \left[\Phi_n^s \circ \Psi_n^{-s} - \mathrm{I}\right] \circ \left(\mathrm{I} + f \circ \Psi_n^{-s}\right) . \tag{4.8.11}$$

The norm of the second term is bounded by $\|f \circ \Psi_n^{-s}\|_n' = \|f\|_n' < 1/2$. By Proposition 4.8.1 and Theorem 4.6.2, we have the bound

$$\left\|\Phi_n^s \circ \Psi_n^{-s} - \mathrm{I}\right\|_{\rho'} \leq \|s(X_n - K_n)\|_\rho \leq C\sigma_n^{m-7-2\alpha-p} r_n \|(\mathbb{I} - \mathbb{P})X\|_\varrho . \tag{4.8.12}$$

provided e.g. that the right hand side of this inequality is bounded by $\rho - \rho'$. This is certainly the case if $m > 7+2\alpha+p$ and $\|X - K\|_\varrho$ is sufficiently small, independently of n. The composition by $\mathrm{I} + f \circ \Psi_n^{-s}$ in equation (4.8.11) is controlled the same way as the composition by $\mathrm{I} + g$ in the proof of Proposition 4.8.2, using also that $\|f \circ \Psi_n^{-s}\|_0 = \|f\|_0$. As a result, the third term on the right hand side of (4.8.11) belongs to $\mathcal{B}_n$ and is bounded in norm by $Ca\sigma_n^{m-7-2\alpha-p}\|X - K\|_\varrho$, which is less than $1/2$ for any $n \geq 1$, if X is sufficiently close to K. Thus, $\|\Phi_n^s \circ F \circ \Psi_n^{-s} - \mathrm{I}\|_n' < 1$, as we wanted to show. **QED**

Assume in the rest of the section that α, m, γ, and a have been chosen in such a way that the hypotheses of Theorem 4.6.2, Proposition 4.8.2 and Proposition 4.8.3 have been satisfied, with $p = 1 + 1/\alpha$. Let $\{F_n\}$, $n \in \mathbb{N}_0$, be a fixed but arbitrary sequence of functions in $\mathcal{A}_0$, such that $F_n \in B_n$, for all $n \geq 0$. Then, we can define

$$\Gamma_{n,m} = (\mathcal{M}_{n+1} \circ \ldots \circ \mathcal{M}_m)(F_m) , \qquad 0 \leq n < m . \tag{4.8.13}$$

Theorem 4.8.4 *Under the above-mentioned assumptions on α, m, γ and a, there exists an open neighborhood B of K in $\mathcal{A}_\varrho$ such that the following holds. For every $X \in B \cap \mathcal{W}$, the limits $\Gamma_n = \lim_{m\to\infty} \Gamma_{n,m}$ exist in $\mathcal{B}_n$, are independent of the choice of the maps F_i, $i \in \mathbb{N}_0$, and satisfy the identities* (4.8.7)*. Furthermore, Γ_0 is an elliptic invariant torus for X, and the map $X \mapsto \Gamma_0$ is analytic and bounded on $B \cap \mathcal{W}$.*

Proof: By Proposition 4.8.2 there exists $N > 0$ such that $\mathcal{M}_n : B_n \to B_{n-1}/2$ contracts distances by a factor of at least $1/2$, if $n \geq N$. Thus, if $N \leq n < m < k$, then the difference $\Gamma_{n,k} - \Gamma_{n,m}$ is bounded in norm by 2^{n-m+1}. This shows that the sequence $m \mapsto \Gamma_{n,m}$ converges in $\mathrm{I} + \mathcal{B}_n$ to a limit Γ_n, and that the limit is independent of the choice of the functions F_m. By choosing $F_m = \Gamma_m$ for all m, we obtain the identities (4.8.7). The analyticity of $X \mapsto \Gamma_0$ follows via chain rule from the analyticity of the maps used in our construction, and from the uniform convergence.

In order to prove that Γ_0 is an invariant torus for X, we will use the identity

$$\Phi^s_{n-1} \circ \mathcal{M}_n(F) \circ \Psi^{-s}_{n-1} = \mathcal{M}_n\big(\Phi_n^{\eta_n s} \circ F \circ \Psi_n^{-\eta_n s}\big) , \tag{4.8.14}$$

which follows from the identity (4.8.1). To be more precise, given $t \in \mathbb{R}$, define $t_n = \lambda_n t$, for all $n \geq 0$. By using that $\lambda_n = \prod_{i=1}^n \eta_i$, together with the bound

$\eta_n \leq \|T_n^{-1}\| < \sigma_n^{-1}$, and the recursion relations (4.6.4) satisfied by σ_n, $n \in \mathbb{N}$, we obtain a bound $\lambda_n \leq C_2^{-1}\sigma_n^{-p}$, for some universal constant $C_2 > 0$ (in fact this is true with $C_2 = 1$). Thus, if $|t| \leq C_2$, then $|t_n| \leq \sigma_n^{-p}$, for all $n \geq 1$. Proposition 4.8.3 now allows us to iterate (4.8.14), and get the identity

$$\Phi_0^t \circ \Gamma_{0,m} \circ \Psi_0^{-t} = (\mathcal{M}_1 \circ \ldots \circ \mathcal{M}_m)(\Phi_m^{t_m} \circ \Psi_m^{-t_m}), \tag{4.8.15}$$

for all $m > 0$. As was shown above, the right hand side of this equation converges in $\mathcal{A}_0$ to Γ_0, and thus the left hand side converges to Γ_0 as well. In addition, $\Gamma_{0,m} \to \Gamma_m$ in $\mathcal{A}_0$, and since convergence in $\mathcal{A}_0$ implies pointwise convergence (see part (1) of Proposition 4.5.2), and the flow Φ_0^t is continuous, we have $\Phi_0^t \circ \Gamma_0 \circ \Psi_0^{-t} = \Gamma_0$. This identity now extends to arbitrary $t \in \mathbb{R}$ by using the group property of the flow, together with the fact that composition with Ψ_0^s is an isometry on $\mathcal{A}_0$.

Finally, notice that by Theorem 4.6.2,

$$\lambda_n \|DX_n\|_\rho \leq C_3 \sigma_n^{m-7-2\alpha-p} r_n \|(\mathbb{I} - \mathbb{P})X_0\|_\varrho, \tag{4.8.16}$$

where C_3 is some universal constant. The left (and thus the right) hand side of this equation is an upper bound on the absolute value of the Lyapunov exponents for the flow of $\lambda_n X_n$ on the the range of Γ_n. Since X_0 is obtained from $\lambda_n X_n$ by a change of coordinates, and Γ_0 is the corresponding invariant torus for X_0, the same upper bound applies to the flow for X_0 on the torus Γ_0. Taking $n \to \infty$ shows that this torus is elliptic. **QED**

4.8.3 Analytic tori and the proof of Lemma 4.1.3

In what follows, the torus Γ_0 associated with $X \in B \cap \mathcal{W}$ will be denoted by Γ_X. The domain parameter ρ used in the introduction is renamed to ϱ', to avoid notational conflicts.

Theorem 4.8.5 *Let $\varrho' > \varrho + \delta$ with $\delta > 0$. Under the same assumptions as in Theorem 4.8.4, the map $X \mapsto \Gamma_X$ defines (via extension) a bounded analytic map from B' to $\mathcal{A}^0_\delta$, where B' is some open neighborhood of K in $\mathcal{A}_{\varrho'}$.*

Proof: For every $u \in \mathbb{R}^d$, define a translation I_u on $\mathbb{C}^d \times \mathbb{C}^\ell$ by setting $I_u(x, y) = (x + u, y)$. If X is a vector field on one of the domains D_r, then $I_u^* X$ denotes the pullback of X under I_u. For functions $F : D_0 \to D_r$ we define $I_u^* F = I_u^{-1} \circ F \circ I_u$. An explicit computation, starting with $\mathcal{T} \circ I_u = I_{Tu} \circ \mathcal{T}$ and $\Lambda_X \circ I_u = I_{Tu} \circ \Lambda_{R^*_{Tu} X}$, shows that the RG transformation $\mathcal{R}$, and the maps $\mathcal{M}_X$ defined in (4.8.2) satisfy

$$\mathcal{R} \circ I_u^* = I_{T^{-1}u}^* \circ \mathcal{R}\,, \qquad \mathcal{M}_{I_u^* X} = I_u^* \circ \mathcal{M}_X \circ (I_{T^{-1}u}^*)^{-1}\,. \tag{4.8.17}$$

Here, we have used that the translations I_u^* are isometries on the spaces $\mathcal{A}_r$, and that the domain of $\mathcal{R}$ is translation invariant (see Definition 4.5.6). This also implies that the manifold $\mathcal{W}$ is invariant under translations I_u^*, which is used in the second identity in (4.8.17).

It is convenient to extend the function $X \mapsto \Gamma_X$ to an open neighborhood of K in $\mathcal{A}_\varrho$ by projecting X onto a point $X' \in \mathcal{W}$ and defining $\Gamma_X = \Gamma_{X'}$. More specifically, we take $X' = (\mathbb{I} + W)((\mathbb{I} - \mathbb{P})X)$, where W is the map defining $\mathcal{W}$, as described in Theorem 4.6.2. If restricted to a sufficiently small open ball $B \subset \mathcal{A}_\varrho$ centered at K, the map $X \mapsto \Gamma_X$ is now analytic and bounded on the whole B.

The construction of Γ_0 in the proof of Theorem 4.8.4, together with the identities (4.8.17), and the invariance property $W = W \circ I_u^*$, shows the identity $\Gamma_{I_u^* X} = I_u^* \Gamma_X$, for all $X \in B$. Thus, if $u \in \mathbb{R}^d$ then

$$\Gamma_X(u,0) = (I_u \circ \Gamma_{I_u^* X})(0,0)\,, \qquad X \in B. \tag{4.8.18}$$

The idea now is to extend the right hand side of (4.8.18) analytically to complex u, by using the analyticity of the map $X \mapsto \Gamma_X$. To this end, choose an open neighborhood B' of K in $\mathcal{A}_{\varrho'}$, such that $I_u^* B' \subset B$, for all $u \in \mathbb{C}^d$ of norm smaller than or equal to $r = \varrho' - \varrho$. Then the right hand side of (4.8.18), regarded as a function of (X, u), is analytic and bounded on the product of B' with the strip $\|\operatorname{Im} u\| < r$. Denoting this function by G, we clearly have $G(X, .) \in \mathcal{A}_\delta^0$ for all $X \in B'$. The analyticity of $X \mapsto G(X, .)$ is obtained e.g. by using a contour integral formula for $(g(t) - g(0) - zg'(0))/t^2$ with $g(t) = G(X + tZ, .)$. **QED**

In the following we prove Lemma 4.1.3 concerning families of vector fields, stated at the beginning of this chapter.

Proof of Lemma 4.1.3: Let $\varrho + \delta < \varrho' < r$. (Recall that ρ has been renamed to ϱ'.) By Theorem 4.6.2 and Theorem 4.8.5, there exists an open ball B in $\mathcal{H}_{\varrho'}$, centered at K, such that $U = \mathbb{P} - W \circ (\mathbb{I} - \mathbb{P})$ defines an analytic map from B to H^u, and that a vector field $X \in B$ has an invariant torus $\Gamma_X \in \mathcal{A}_\delta^0$ with frequency vector ω, whenever $U(X) = 0$. Furthermore, $X \mapsto \Gamma_X$ is analytic on $B \cap \mathcal{W}$.

By our non-degeneracy assumptions, the function G_ε,

$$G_\varepsilon(z, v, s) = zK + (1+z) J_v^*(\varepsilon Z + s)\,, \tag{4.8.19}$$

defines a diffeomorphism $\phi = \mathbb{P} \circ G_\varepsilon$ between open neighborhoods of the origin in the spaces $\mathbb{C} \oplus V \oplus H_0^u$ and H^u. Let b be an open ball in H^u of radius smaller than $R/2$, centered at the origin, where R is the radius of B. If $\varepsilon > 0$ and b are chosen sufficiently small, then $G'_\varepsilon = (\mathbb{I} - \mathbb{P})(G_\varepsilon \circ \phi^{-1})$ is a family in $\mathcal{F}(b)$ of norm smaller than $R/2$, and the equation $F(\phi(z, v, s)) = (1 + z)J_v^* f(s)$ defines an analytic map $f \mapsto F$ from some open neighborhood B_2 of f_ε in $\mathcal{F}(b_2)$, into $\mathcal{F}(b)$. The image of f_ε under this map is the family F_ε given by $F_\varepsilon(\sigma) = K + \sigma + G'_\varepsilon(\sigma)$. By using that $\mathbb{P} \circ G'_\varepsilon = 0$ and $W \circ G'_\varepsilon = K$, we see that $U \circ F_\varepsilon$ is the identity map on b. Thus, by the implicit function theorem, the equation $(U \circ F)(\sigma) = 0$ has a unique solution $\sigma = \sigma_F$, for any family F sufficiently close to F_ε in $\mathcal{F}(b)$, and this solution depends analytically on F. The assertion now follows, with $(z_f, v_f, s_f) = \phi^{-1}(\sigma_F)$ and $c_f = (1 + z_f)$. **QED**

Chapter 5

Renormalization of vector fields for Brjuno frequencies

In this chapter, we construct a renormalization scheme for vector fields that applies to the problem of persistence of invariant tori with Brjuno frequency vectors.

5.1 Introduction and main results

As mentioned in Chapter 2, a fundamental problem of KAM theory is to state the weakest possible condition on the frequency vectors for the preservation of quasiperiodic motion in near integrable systems. The answer to this question is known only in rare cases, such as the Siegel problem [77] and circle diffeomorphisms [78], where the condition is Brjuno. As far as the construction of smooth invariant tori is concerned, the work from the previous chapters covers the classical KAM results, but not the later extensions to Brjuno type frequency vectors [5, 7, 8, 19, 31, 32, 69]. This is due to the fact that the current approach requires good bounds on a continued fractions expansion, such as the ones obtained in [41, 42] for Diophantine frequency vectors. Unfortunately, there seem to be significant obstacles to obtaining such bounds for Brjuno vectors in dimensions $d > 2$.

This has motivated us to develop a renormalization scheme that does not rely on continued fractions. As it turns out, it applies quite naturally to rotation vectors $\omega \in \mathbb{R}^d$ that satisfy the following Brjuno condition [7, 8]:

$$\sum_{n=1}^{\infty} 2^{-n} \ln(1/\Omega_n) < \infty \,, \qquad \Omega_n = \min_{0<|\nu|\le 2^n} |\omega \cdot \nu| \,, \tag{5.1.1}$$

where ν denotes lattice points in $\mathbb{Z}^d$.

We define the renormalization operator on a Banach space of vector fields analytic on a complex neighborhood of $\mathbb{T}^d \times \{0\}$, with $0 \in \mathbb{R}^m$, $d \ge 2$ and $m \ge 0$, as

$$\mathcal{R}(X) = \eta^{-1} \mathcal{T}_\mu^* \mathcal{U}_X^* X \,, \tag{5.1.2}$$

where $\mathcal{U}_X$ is some change of coordinates designed to bring the renormalized vector field into some appropriate normal form (Figure 5.1), $\mathcal{T}_\mu^*$ is the pullback of the map

$$\mathcal{T}_\mu(x, y) = (Tx, \mu \bar{T} y) \,, \tag{5.1.3}$$

and η is a positive constant. Here, $\bar{T}$ denotes the $m \times m$ identity matrix (or the inverse of $T^\star$, if desired for the renormalization of Hamiltonian vector fields, where $m = d$). There is no natural choice for the scaling parameters η, but since all members of the family $\eta \mapsto \eta^{-1} \mathcal{T}_\mu^* X$ are equivalent, in the sense that they yield the same frequency ratios, it is useful to choose $\eta = \eta(X)$ in such a way that $\mathcal{R}(X)$ becomes a specific (normalized) representative of the family. This ensures contraction within this family of equivalent systems. The same considerations apply in principle to the choice of μ, but for the vector fields considered here, it suffices to choose for μ a small positive constant that makes $\mu\bar{T}$ a contraction.

The idea pursued here is to avoid the integer approximation and renormalize directly with real matrices. Since there is no longer any reason to stay with $\mathrm{SL}(d,\mathbb{R})$, we choose the matrix T in the Definition (5.1.3) of the scaling $\mathcal{T}_\mu$ to be of the general form

$$T(x) = \eta^{-1} x_{\|} + \beta x_{\perp}\,, \qquad 0 < \eta\,, \beta < 1\,, \tag{5.1.4}$$

where $x = x_{\|} + x_{\perp}$ is the decomposition of $x \in \mathbb{R}^d$ into a component $x_{\|}$ parallel to ω, and a component $x_{\perp}$ perpendicular to ω. Notice that, if $d = 2$ and $\omega_1/\omega_2 = 1/(k + 1/(k + \ldots))$, then the choice $\eta = \beta = (\omega_1/\omega_2)^2$ makes T in fact an integer matrix. A matrix of the type (5.1.4) will be referred to as a *scaling matrix*. The corresponding RG transformation $\mathcal{R}$ is taken to be again of the form (5.1.2), with η^{-1} the expanding eigenvalue of T. Our choice of μ and $\mathcal{U}_X$ will be described later. Clearly, $K = (\omega, 0)$ is a fixed point for $\mathcal{R}$. We note that, by choosing the time scaling η^{-1} in (5.1.2) to be the same as the spatial scaling η^{-1} in (5.1.4), which is independent of the vector field X, we allow $\mathcal{R}$ to have a non-contracting direction. However, this direction is trivial and can be taken care of later.

With this choice of scaling T, it becomes necessary to consider toral domains of the form $\mathbb{T}^d = \mathbb{R}^d/\mathcal{Z}$, where $\mathcal{Z}$ is a simple lattice in $\mathbb{R}^d$. Functions on such a torus can be identified with functions on $\mathbb{R}^d$ that are invariant under $\mathcal{Z}$-translations, or equivalently, with quasiperiodic functions on $\mathbb{R}^d$ whose frequency module lies in the dual lattice (the set of points $v \in \mathbb{R}^d$ satisfying $\exp(iv \cdot z) = 1$, for all $z \in \mathcal{Z}$). For convenience, we will now perform a linear change of coordinates in $\mathbb{R}^d$, such that $\omega = (1, 0, \ldots, 0)$. The lattice obtained from $2\pi\mathbb{Z}^d$ under this transformation will be denoted by $\mathcal{Z}_0$, and its dual lattice by $\mathcal{V}_0$. The frequencies ν in (5.1.1) now range over $\mathcal{V}_0$.

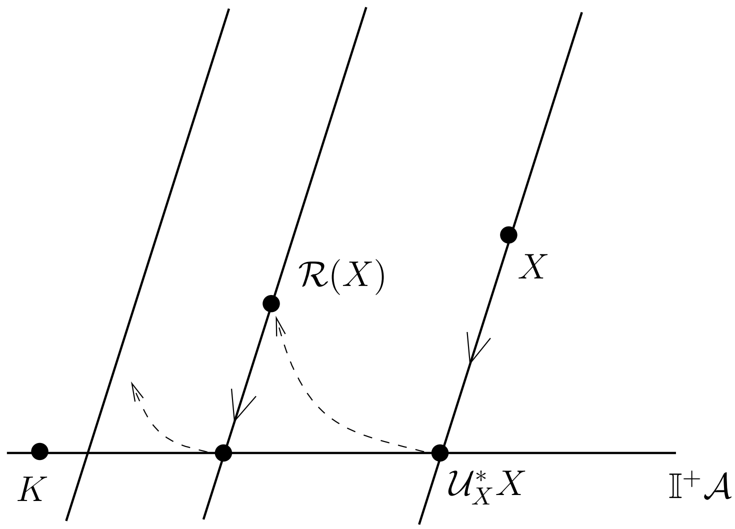

Figure 5.1: *The action of the renormalization operator on a vector field on $\mathcal{W}$. Vector fields in normal form belong to $\mathbb{I}^+\mathcal{A}$.*

Our analysis applies to vector fields that are close to $K = (\omega, 0)$. We assume analyticity on a complex neighborhood D_ϱ of $D_0 = \mathbb{T}^d \times \{0\}$, characterized by the conditions $|\mathrm{Im}\, x_i| < \varrho$ and $|y_j| < \varrho$. Denote by Φ_X the flow for a vector field X. An invariant torus for X, with frequency vector ω, is a continuous embedding Γ of D_0 into the domain of X, such that $\Gamma \circ \Phi_K^t = \Phi_X^t \circ \Gamma$. Denote by A^u the space of all vector fields $Y(x, y) = (w, My + v)$, with (w, v) a vector in $\mathbb{C}^d \times \mathbb{C}^m$ and M a complex $m \times m$ matrix. In Section 5.2, we will introduce Banach spaces $\mathcal{A}_\varrho(\mathcal{V})$ of analytic vector fields on D_ϱ, having frequency module in $\mathcal{V}$, and a projection operator $\mathbb{P}$ from $\mathcal{A}_\varrho(\mathcal{V})$ onto A^u. The subspace of functions in $\mathcal{A}_\varrho(\mathcal{V})$, that do not depend on the coordinate $y \in \mathbb{C}^m$, will be denoted by $\mathcal{A}_\varrho^0(\mathcal{V})$. A function will be called "real" if it takes real values for real arguments. Our main result is the following.

Theorem 5.1.1 *Assume that ω satisfies the Brjuno condition* (5.1.1). *Then there exists a sequence of scaling matrices T_n, and a corresponding sequence*

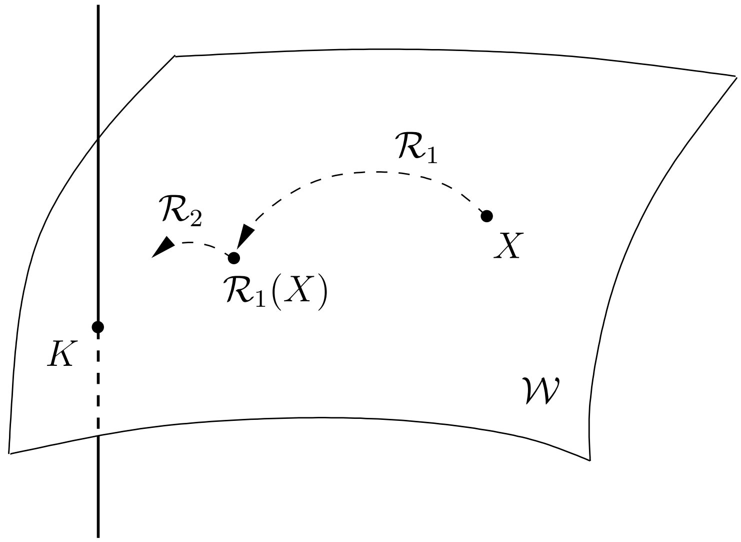

Figure 5.2: *The space of vector fields and renormalization of a vector field X on the stable manifold $\mathcal{W}$.*

of RG transformations $\mathcal{R}_n$ of the form (5.1.2)*, such that the following holds. Define $\mathcal{V}_n = T_n \mathcal{V}_{n-1}$ for $n \in \mathbb{N}$. Then $\mathcal{R}_n$ is an analytic map, from some open neighborhood $\mathcal{D}_{n-1}$ of K in $\mathcal{A}_\varrho(\mathcal{V}_{n-1})$, to $\mathcal{A}_\varrho(\mathcal{V}_n)$. The set $\mathcal{W}$ of infinitely renormalizable vector fields X_0 in $\mathcal{D}_0$, characterized by the property that $X_n = \mathcal{R}_n(X_{n-1})$ belongs to $\mathcal{D}_n$ for $n \in \mathbb{N}$, is the graph of an analytic function $W : (\mathbb{I} - \mathbb{P})\mathcal{D}_0 \to \mathbb{P}\mathcal{D}_0$, satisfying $W(0) = K$ and $DW(0) = 0$. If $\rho > \varrho + \delta$, with $\delta > 0$, then every vector field $X \in \mathcal{W} \cap \mathcal{A}_\rho(\mathcal{V}_0)$ has an elliptic invariant torus $\Gamma_X \in \mathcal{A}^0_\delta(\mathcal{V}_0)$ with frequency vector ω. The map $X \mapsto \Gamma_X$ is real analytic on $\mathcal{W} \cap \mathcal{A}_\rho(\mathcal{V}_0)$.*

The bounds obtained in the proof of this theorem are uniform within classes of Brjuno vectors $\mathcal{B}(\Omega')$ described at the end of Section 5.3.

In addition, we obtain results analogous to those in the previous chapter, concerning the restriction of W to special types of vector fields (Hamiltonian, reversible, divergence free, symmetric) and the reduction of the number of parameters via nondegeneracy conditions. Since the proofs are completely analogous as well, we refer to the previous chapter for details.

The main new aspect in this chapter is the choice of the scaling matrices T_n, and the control of the corresponding sequence of RG transformations $\mathcal{R}_n$. The choice of the coordinate change $\mathcal{U}_X$ is determined by the same considerations as in the previous chapter. Its role is to compensate for the loss of analyticity that would result from the scaling $X \mapsto \mathcal{T}_\mu^* X$. In this step, we use a normal form theorem proved in Section 4.4. Thus, controlling a single RG step is quite simple (see Section 5.2). In Section 5.3, we define the matrices T_n and give estimates on the transformations $\mathcal{R}_n$. Then we apply a stable manifold theorem for sequences of maps from Section 4.7 to obtain the manifold $\mathcal{W}$ described in Theorem 5.1.1. The construction of invariant tori for vector fields $X \in \mathcal{W}$ is described in Section 5.4.

5.2 A single renormalization step

5.2.1 The spaces of vector fields

As mentioned in the introduction, we work in coordinates where the frequency vector is $\omega = (1, 0, \ldots, 0)$. The torus considered in this section is $\mathbb{T}^d = \mathbb{R}^d/\mathcal{Z}$, where $\mathcal{Z}$ is some simple lattice in $\mathbb{R}^d$. The dual lattice will be denoted by $\mathcal{V}$.

Unless specified otherwise, our norm on $\mathbb{C}^n$ is $\|v\| = \sup_j |v_j|$. Another norm that will be used is $|v| = \sum_j |v_j|$. For linear operators between normed linear spaces, we will always use the operator norm, unless stated otherwise. Denote by D_ρ the set of all vectors (x, y) in $\mathbb{C}^d \times \mathbb{C}^m$ characterized by $\|\mathrm{Im}\, x\| < \rho$ and $\|y\| < \rho$. Define $\mathcal{A}_\rho(\mathcal{V})$ to be the space of all analytic vector field X on D_ρ, with frequency module in $\mathcal{V}$, and with a finite norm

$$\|X\|_\rho = \sum_{v,\alpha} \|X_{v,\alpha}\| e^{\rho|v|} \rho^{|\alpha|}\,, \qquad X(x,y) = \sum_{v,\alpha} X_{v,\alpha} e^{iv\cdot x} y^\alpha\,, \tag{5.2.1}$$

where $v \cdot x = \sum_j v_j x_j$ and $y^\alpha = \prod_j y_j^{\alpha_j}$. The sums in this equation range over all $v \in \mathcal{V}$ and $\alpha \in \mathbb{N}^m$. In Section 5.4, we will also use functions with domain $D_0 = \mathbb{T}^d \times \{0\}$. Denote by $\mathcal{A}_0(\mathcal{V})$ the Banach space of continuous functions $F : D_0 \to \mathbb{C}^{d+m}$, with frequency module in $\mathcal{V}$, for which the norm $\|F\|_0 = \sum_v \|F_v\|$ is finite, where $\{F_v\}$ are the Fourier coefficients of $F = \sum_v F_v e^{iv\cdot x}$. Since the lattice $\mathcal{V}$ is fixed in this section, we will simply write $\mathcal{A}_\rho$ in place of $\mathcal{A}_\rho(\mathcal{V})$.

Proposition 5.2.1 *Let $X \in \mathcal{A}_\rho$ and $Z \in \mathcal{A}_{\rho'}$, with $0 \le \rho' \le \rho$. Then*

(i) $\|X(x,y)\| \le \|X\|_\rho$ *for all* $(x,y) \in D_\rho$.

(ii) $(DX)Z \in \mathcal{A}_{\rho'}$ *and* $\|(DX)Z\|_{\rho'} \le (\rho - \rho')^{-1}\|X\|_\rho \|Z\|_{\rho'}$, *if* $\rho' < \rho$.

(iii) $X \circ (\mathrm{I} + Z) \in \mathcal{A}_{\rho'}$ *and* $\|X \circ (\mathrm{I} + Z)\|_{\rho'} \le \|X\|_\rho$, *if* $\rho' + \|Z\|_{\rho'} \le \rho$.

The proof of these estimates is straightforward and will be omitted. In what follows, we always assume that $\rho > 0$, unless specified otherwise.

5.2.2 Resonant and non-resonant modes

We assume that the components of ω are rationally independent with respect to $\mathcal{V}$, in the sense that the first component $v_{||}$ of any nonzero vector $v \in \mathcal{V}$ is nonzero. Then, given any $L \ge 1$, we can find $\ell > 0$ such that

$$|v_\perp| > L \quad \text{or} \quad |v_{||}| \ge \ell\,, \qquad \forall v \in \mathcal{V} \setminus \{0\}\,. \tag{5.2.2}$$

In other words, all points in $\mathcal{V}$, except for the origin, lie outside the rectangle $|v_\perp| \leq L$ and $|v_{||}| < \ell$. Notice that the scaling (5.1.4) shrinks the length L of the excluded rectangle, and expands its width ℓ. In what follows, the parameters L, ℓ, η, β are assumed to be given, subject to the conditions (5.1.4) and (5.2.2).

Definition 5.2.2 *Denote by S the generator of the one-parameter group of scalings $\mu \mapsto \mathcal{S}_\mu^*$, defined by $\mathcal{S}_\mu(x, y) = (x, \mu y)$. Given any subset J of $I = \mathcal{V} \times \{-1, 0, 1, 2, \ldots\}$, define $P(J)$ to be the joint spectral projection in $\mathcal{A}_\rho(\mathcal{V})$ for the operators $(-i\nabla_x, S)$, associated with the eigenvalues (v, k) in J. Let $\tau = (1+\beta)/2$. Given $\gamma \geq 1$ to be chosen later, let I^+ be the set of pairs $(v, k) \in I$ satisfying $|Tv| \leq \tau|v|$ or $|Tv| \leq \tau(k - \gamma)$, and let I^- be the complement of I^+ in I. Define $\mathbb{I}^\pm = P(I^\pm)$. The resonant and nonresonant parts of a vector field $X \in \mathcal{A}_\rho$ are defined as $\mathbb{I}^+X$ and $\mathbb{I}^-X$, respectively. In addition, we define $\mathbb{E}_k = P(\{(0, k)\})$, for each $k \geq -1$. The torus averaging operator is then given by $\mathbb{E} = \sum_k \mathbb{E}_k$.*

As we will see later, the scaling $X \mapsto \mathcal{T}_\mu^* X$ is well behaved when restricted to resonant vector fields. Thus, before applying this scaling, we try to perform a change of variables $X \mapsto \mathcal{U}_X^* X$ that eliminates the nonresonant part of X. The normal form theorem of Section 4.4 shows that this is possible, provided that the problem can be solved to first order in the size of $X - K$. The equation for the map $\mathcal{U}_X$, and for the vector field $Z = \mathbb{I}^- Z$ generating its first order approximation Φ_Z^1, is

$$\mathbb{I}^-(X + [Z, X]) = 0\,, \qquad \mathbb{I}^-\mathcal{U}_X^* X = 0\,, \tag{5.2.3}$$

where $[Z, X] = (DX)Z - (DZ)X$. The following proposition is used to solve the first part of this equation. Given any positive real number r, denote by $\mathcal{A}'_r$ the set of vector fields $X \in \mathcal{A}_r$ whose first partial derivatives belong to $\mathcal{A}_r$. Assume that

$$2\sigma L < \ell\,, \qquad \sigma = \frac{1}{2}(1-\beta)\eta\,. \tag{5.2.4}$$

Proposition 5.2.3 *If $r > 0$ and $Z \in \mathcal{A}'_r$ is nonresonant, then*

$$\|[Z,K]\|_r \geq \sigma\|Z\|_r\,, \qquad \|[Z,K]\|_r \geq \frac{\sigma r}{\sigma r + r + \gamma + 2}\|DZ\|_r\,. \tag{5.2.5}$$

Proof: Assume that (v,k) belongs to I^-. In particular, we have $|Tv| > \tau|v|$, or equivalently, $\eta^{-1}|v_{||}| + \beta|v_\perp| > \tau|v_{||}| + \tau|v_\perp|$. This immediately implies that $|v_{||}| > \sigma|v_\perp|$. Combining this with the condition $|Tv| > \tau(k-\gamma)$, we also have

$$\sigma^{-1}|v_{||}| = \tau^{-1}(\eta^{-1}|v_{||}| + \beta\sigma^{-1}|v_{||}|) > \tau^{-1}(\eta^{-1}|v_{||}| + \beta|v_\perp|) = \tau^{-1}|Tv| > k - \gamma \tag{5.2.6}$$

The inequality $|v_{||}| > \sigma|v_\perp|$, together with (5.2.2) and (5.2.4), also implies that $|v_{||}| > \sigma$. These bounds show that if $Z \in \mathbb{I}^-\mathcal{A}'_r$ and $Y = [Z,K]$, then $\|Z\|_r \leq \sigma^{-1}\|Y\|_r$ and

$$\sum_{j=2}^{d}\left\|\frac{\partial}{\partial x_j}Z\right\|_r \leq \frac{1}{\sigma}\|Y\|_r\,, \qquad \sum_{j=1}^{m}\left\|\frac{\partial}{\partial y_j}Z\right\|_r \leq \frac{\gamma+2}{\sigma r}\|Y\|_r\,. \tag{5.2.7}$$

As a result we obtain (5.2.5). **QED**

This proposition, together with Proposition 5.2.1, allows us to apply the normal form theorem of Section 4.4, which directly implies the following lemma. Let $\varrho > 0$ be fixed once and for all. What we will call a universal constant may depend on the choice of ϱ, but not on any other parameter.

Lemma 5.2.4 *There exist universal constants C_1 and C_2, such that the following holds. Let $\rho' > 0$ and $\rho \geq \rho' + \sigma\varrho$. If X is any vector field in $\mathcal{A}'_\rho$,* *satisfying*

$$\|X-K\|'_\rho \le C_1(\sigma/\gamma)\,, \qquad \|\mathbb{I}^- X\|_\rho \le C_1(\sigma/\gamma)^2\,, \tag{5.2.8}$$

then there exists a vector field $Z \in \mathbb{I}^- \mathcal{A}_\rho$ *and a change of coordinates* $\mathcal{U}_X : D_{\rho'} \to D_\rho$ *, solving equation* (5.2.3)*. The vector field* $\mathcal{U}_X^* X$ *belongs to* $\mathcal{A}_{\rho'}$ *, and*

$$\begin{aligned} \|Z\|_\rho\,, \|\mathcal{U}_X - \mathrm{I}\|_{\rho'} &\le C_2(\gamma/\sigma)\|\mathbb{I}^- X\|_\rho\,, \\ \|\mathcal{U}_X^* X - X\|_{\rho'} &\le C_2(\rho-\rho')^{-1}(\gamma/\sigma)\|\mathbb{I}^- X\|_\rho\,, \\ \|\mathcal{U}_X^* X - X - [Z,X]\|_{\rho'} &\le C_2(\rho-\rho')^{-3}(\gamma/\sigma)^3\|\mathbb{I}^- X\|_\rho^2\,. \end{aligned} \tag{5.2.9}$$

The map $X \mapsto \mathcal{U}_X$ *is continuous in the region defined by* (5.2.8)*, and analytic in its interior.*

Next, we assume that the scaling parameters η, β and μ satisfy

$$\eta < 1/2\,, \qquad e^{-\varrho\frac{(1-\beta)}{6}L} \le (4\mu)^{\gamma+1}\,, \qquad 4\mu \le e^{-\varrho}\,. \tag{5.2.10}$$

Lemma 5.2.5 *If* $\varrho(2+\beta)/3 \le \rho' \le \varrho$*, then* $\mathcal{T}_\mu^*$ *defines a bounded linear operator from* $\mathbb{I}^+\mathcal{A}_{\rho'}(\mathcal{V})$ *to* $\mathcal{A}_\varrho(T\mathcal{V})$*, with the property that*

$$\begin{aligned} \|\mathcal{T}_\mu^* \mathbb{E}_k X\|_\varrho &\le 8\eta^{-1}(4\mu)^k \|\mathbb{E}_k X\|_{\rho'}\,, \\ \|\mathcal{T}_\mu^* \mathbb{I}^+(\mathbb{I}-\mathbb{E})X\|_\varrho &\le 2\eta^{-1}(4e^\varrho\mu)^\gamma \|\mathbb{I}^+(\mathbb{I}-\mathbb{E})X\|_{\rho'}\,. \end{aligned}$$

Proof: By our choice of norm (5.2.1), it suffices to verify the given bounds for vector fields $X = P(J)Y$, with $J \subset I^+$ containing a single point, say $J = \{(v,k)\}$.

Let $b = \varrho/(\rho'\beta)$. Then it follows essentially from the definitions that

$$\|\mathcal{T}_\mu^* P(J)Y\|_\varrho \leq 2\eta^{-1} e^A \|P(J)Y\|_{\rho'}\,, \qquad A = \varrho|Tv| - \rho'|v| + k\ln(b\mu)\,. \tag{5.2.11}$$

Setting $v = 0$, and using that $1 < b < 4$, yields the first bound in (5.2.11).

In order to prove the second bound, assume that (v, k) belongs to I^+, and that $v \neq 0$. Consider first the case $|Tv| \leq \tau|v|$. It leads to $|v_{\|}| < 2\sigma|v_\perp|$, if we use that $\eta\tau < 1/2$. This inequality excludes frequencies v that satisfy $|v_\perp| \leq L$ and $|v_{\|}| \geq \ell$, due to the condition (5.2.4). Thus, we must have $|v_\perp| > L$ by condition (5.2.2). Consequently,

$$A \leq -\varrho\Big(\frac{\rho'}{\varrho} - \tau\Big)|v| + k\ln(b\mu) \leq -\varrho\frac{1-\beta}{6}L + k\ln(b\mu)\,, \tag{5.2.12}$$

and the second bound in (5.2.11) follows by using (5.2.10).

Next, consider the case $|Tv| \leq \tau(k - \gamma)$. Notice that $k > \gamma$ here, since v is nonzero. By using the bound $A \leq \varrho(k - \gamma) + k\ln(b\mu)$, together with (5.2.11), we obtain

$$\|\mathcal{T}_\mu^* P(J)Y\|_\varrho \leq 2\eta^{-1}\big(be^\varrho\mu\big)^k \|P(J)Y\|_{\rho'}\,. \tag{5.2.13}$$

This again implies the second bound in (5.2.11). **QED**

5.2.3 Estimates for a single renormalization step

Combining the preceding two lemmas, we obtain the following theorem. Notice that, by property (5.2.9), the restriction of $\mathcal{R}$ to the subspace $\mathbb{P}\mathcal{A}_\varrho(\mathcal{V})$ defines a linear operator from $\mathbb{P}\mathcal{A}_\varrho(\mathcal{V})$ to $\mathbb{P}\mathcal{A}_\varrho(T\mathcal{V})$. This operator will be denoted by $\mathcal{L}$.

Theorem 5.2.6 *There exist universal constants $C, R > 0$, such that the follow-*

ing holds, under the given assumptions on $L, \ell, \eta, \beta, \gamma$ and μ. Let B be the open ball in $\mathcal{A}_\varrho(\mathcal{V})$ of radius $R(\sigma/\gamma)^2$, centered at K. Then $\mathcal{R}$ is a bounded analytic map from B to $\mathcal{A}_\varrho(T\mathcal{V})$, satisfying $\|\mathcal{L}^{-1}\| \leq 1$ and

$$\begin{aligned} \|(\mathbb{I}-\mathbb{E})\mathcal{R}(X)\|_\varrho &\leq \eta^{-2}(1-\beta)^{-1}(\gamma/\sigma)(C\mu)^\gamma \|(\mathbb{I}-\mathbb{E})X\|_\varrho\,, \\ \|(\mathbb{I}-\mathbb{P})\mathcal{R}(X)\|_\varrho &\leq C\eta^{-2}(1-\beta)^{-1}(\gamma/\sigma)\mu\|(\mathbb{I}-\mathbb{P})X\|_\varrho\,, \\ \|\mathbb{E}\mathcal{R}(X)-\mathcal{R}(\mathbb{E}X)\|_\varrho &\leq C\eta^{-2}(1-\beta)^{-3}(\gamma/\sigma)^3\mu^{-1}\|(\mathbb{I}-\mathbb{E})X\|_\varrho^2\,. \end{aligned} \tag{5.2.14}$$

Proof: Let $\rho = \varrho - \varrho(1-\beta)/12$ and $\rho' = \rho - \varrho(1-\beta)/4$. Then there exists a universal constant $R > 0$, such that the conditions (5.2.8) in Lemma 5.2.4 hold, whenever X belongs to the domain B, defined by $\|X-K\|_\varrho < R(\sigma/\gamma)^2$. Here, we have used that $\sigma < (1-\beta)/4$.

By Lemma 5.2.5, we have

$$\begin{aligned} \|(\mathbb{I}-\mathbb{E})\mathcal{R}(X)\|_\varrho &= \eta^{-1}\|\mathcal{T}_\mu^*(\mathbb{I}-\mathbb{E})\mathcal{U}_X^* X\|_\varrho \\ &\leq 2\eta^{-2}\left(4e^\rho\mu\right)^\gamma\left[\|(\mathbb{I}-\mathbb{E})X\|_{\rho'} + \|\mathcal{U}_X^*X - X\|_{\rho'}\right]. \end{aligned} \tag{5.2.15}$$

Using the bound in (5.2.9) on the norm of $\mathcal{U}_X^*X - X$, together with the fact that $\mathbb{I}^{-}\mathbb{E} = 0$, we obtain the first inequality in (5.2.14). Similarly, Lemma 5.2.5 implies that

$$\|\mathbb{E}_k\mathcal{R}(X)\|_\varrho \leq C_1\eta^{-2}\mu\left[\|\mathbb{E}_kX\|_{\rho'} + \|\mathbb{E}_k(\mathcal{U}_X^*X - X)\|_{\rho'}\right], \tag{5.2.16}$$

for all $k \geq 1$. Here, and in what follows, $C_1, C_2, \ldots$ denote positive universal constants. Summing over $k \geq 1$ to get a bound on $\|(\mathbb{E}-\mathbb{P})\mathcal{R}(X)\|_\varrho$, and then adding (5.2.15), yields a bound analogous to (5.2.16), but with $\mathbb{E}_k$ replaced by $\mathbb{I}-\mathbb{P}$. Using again the bound in (5.2.9) on $\mathcal{U}_X^*X - X$, and the fact that $\mathbb{I}^{-}\mathbb{P} = 0$, we obtain the second inequality in (5.2.14).

By Lemma 5.2.5, we also have a bound

$$\begin{aligned}
\|\mathbb{E}\mathcal{R}(X) - \mathcal{R}(\mathbb{E}X)\|_\varrho &= \eta^{-1}\|\mathcal{T}_\mu^*\mathbb{E}(\mathcal{U}_X^* X - X)\|_\varrho \\
&\le 2\eta^{-2}\mu^{-1}\|\mathbb{E}(\mathcal{U}_X^* X - X)\|_{\rho'} . \qquad (5.2.17)
\end{aligned}$$

Using Lemma 5.2.4, the norm on the right hand side of (5.2.17) can be estimated as follows:

$$\|\mathbb{E}(\mathcal{U}_X^* X - X)\|_{\rho'} \le C_2(1-\beta)^{-3}(\gamma/\sigma)^3\|(\mathbb{I}-\mathbb{E})X\|_\rho^2 + \|\mathbb{E}[Z, X]\|_{\rho'} , \qquad (5.2.18)$$

where $Z = \mathbb{I}^{-} Z$ is the vector field described in . Since $\mathbb{E}Z = 0$, we have $\mathbb{E}[Z, \mathbb{E}X] = 0$. As a result,

$$\begin{aligned}
\|\mathbb{E}[Z, X]\|_{\rho'} &= \|\mathbb{E}[Z, (\mathbb{I}-\mathbb{E})X]\|_{\rho'} \le C_3(1-\beta)^{-1}\|Z\|_\rho\|(\mathbb{I}-\mathbb{E})X\|_\rho \\
&\le C_4(1-\beta)^{-1}(\gamma/\sigma)\|(\mathbb{I}-\mathbb{E})X\|_\rho^2 . \qquad (5.2.19)
\end{aligned}$$

Here, we have used Proposition 5.2.1 and the bound on $\|Z\|_\rho$ from Lemma 5.2.4. Combining the last three equations yields the third inequality in (5.2.14).

In order to bound the inverse of $\mathcal{L}$, let Y be a vector field in $\mathbb{P}\mathcal{A}_\rho$. Then Y can be written as $Y(x, y) = (w, My + v)$, and the last inequality in (5.2.14) now follows from the fact that

$$(\mathcal{L}^{-1}Y)(x, y) = \eta(Tw, My + \mu v) . \qquad (5.2.20)$$

Here, we have used that $\bar{T} = \mathrm{I}$, except (optionally) for the renormalization of purely Hamiltonian vector fields, where M and v are zero. **QED**

5.3 Iterated renormalization group transformations

Let now $\mathcal{V}_0$ be a simple lattice in $\mathbb{R}^d$, such that the Brjuno condition (5.1.1) holds if the frequencies ν are chosen from $\mathcal{V}_0$. Using the same set of frequencies, define

$$a_n = \sum_{k=n}^{\infty} 2^{n-k}\Big[2^{-k-\kappa}\ln(1/\Omega'_{k+\kappa}) + (k+\kappa')^{-2}\Big]\,, \qquad \Omega'_n = \min_{0<|\nu_\perp|<2^n} |\nu_\parallel|\,, \tag{5.3.1}$$

for all positive integers n. Here $\kappa, \kappa' > 2$ are two integer constants to be determined later. Then the Brjuno condition (5.1.1) is equivalent to the condition that the resulting sequence $\{a_n\}$ is summable. We remark that the weighted sum has been included in the Definition (5.3.1) in order to limit the local growth of the sequence $\{a_n\}$. And the term $(k+\kappa')^{-2}$ has been included to avoid sequences $\{a_n\}$ that decrease too rapidly. This allows for a more uniform treatment of all Brjuno vectors.

Define $\lambda_0 = 1$ and

$$\lambda_n = 2^{-n-\kappa} e^{-2^{n+\kappa} a_n}\,, \quad \eta_n = \frac{\lambda_n}{\lambda_{n-1}}\,, \quad A_n = \sum_{k=n}^{\infty} a_k\,, \quad \beta_n = \frac{A_{n+1}}{A_n}\,, \tag{5.3.2}$$

for all positive integers n. Consider the corresponding scaling transformations

$$P_n(x) = \lambda_n^{-1} x_\parallel + A_1^{-1} A_{n+1} x_\perp\,, \qquad T_n(x) = \eta_n^{-1} x_\parallel + \beta_n x_\perp\,. \tag{5.3.3}$$

Notice that $P_n = T_1 T_2 \cdots T_n$ by equation (5.3.2). These quantities, will now be used to define the n-th step RG transformation $\mathcal{R} = \mathcal{R}_n$. To this end, we need to verify the assumptions made in Section 5.2. Clearly, $\beta = \beta_n$ is positive and less than one, since $n \mapsto A_n$ is a decreasing sequence. And the condition on $\eta = \eta_n$ in equation (5.2.10) follows from the fact that $a_n > a_{n-1}/2$ for $n > 1$, and that $\lambda_1 < 1/2$.

The geometric data $\mathcal{V}$, L and ℓ used in step n are

$$\mathcal{V}_{n-1} = P_{n-1}\mathcal{V}_0\,, \qquad L_{n-1} = A_1^{-1}A_n 2^{n+\kappa}\,, \qquad \ell_{n-1} = 2^{n+\kappa}\eta_n\,. \tag{5.3.4}$$

These definitions immediately imply (5.2.4). The following proposition shows that the condition (5.2.2) holds for all $v \in \mathcal{V}$.

Proposition 5.3.1 *If $v \in \mathcal{V}_{n-1}$ is nonzero, then either $|v_{\|}| \geq \ell_{n-1}$ or $|v_{\perp}| > L_{n-1}$.*

Proof: Assume that $v \in \mathcal{V}_{n-1}$ satisfies $0 < |v_{\perp}| \leq L_{n-1}$. Then the corresponding lattice point $\nu = P_{n-1}^{-1}v$ in $\mathcal{V}_0$ satisfies $|\nu_{\perp}| \leq A_1 A_n^{-1} L_{n-1} = 2^{n+\kappa}$, and thus $|\nu_{\|}| \geq \Omega'_{n+\kappa}$ by (5.3.1). Since we have $\lambda_n < 2^{-n-\kappa}\Omega'_{n+\kappa}$, this yields

$$|v_{\|}| = \lambda_{n-1}^{-1}|\nu_{\|}| \geq \eta_n \lambda_n^{-1}\Omega'_{n+\kappa} > \eta_n 2^{n+\kappa} = \ell_{n-1}\,, \tag{5.3.5}$$

as claimed. **QED**

The second condition in (5.2.10) is satisfied simply by choosing $\mu = \mu_n$, where

$$\mu_k = \exp\Big\{-\frac{\varrho}{6}\cdot\frac{1-\beta_k}{\gamma+1}L_{k-1}\Big\} = \exp\Big\{-\frac{\varrho}{6(\gamma+1)A_1}\cdot 2^{k+\kappa}a_k\Big\}\,, \qquad k \geq 1\,. \tag{5.3.6}$$

Finally, the third inequality in (5.2.10) is taken care of by choosing κ' and κ sufficiently large, as the following proposition shows.

Proposition 5.3.2 *For all $k \geq 1$, $\mu_{k+1} < \mu_k < \mu_{k+1}^{1/4}$. Furthermore, given $\gamma \geq 1$ and $C, N > 0$, if κ' and then κ are chosen sufficiently large, then for all $k \geq 1$,*

$$\mu_k \leq Ce^{-N2^{k+\kappa}a_k}\,, \qquad \mu_k \leq C\eta_k^N\,, \qquad \mu_k \leq C(1-\beta_k)^N\,. \tag{5.3.7}$$

Proof: The inequality $\mu_{k+1} < \mu_k < \mu_{k+1}^{1/4}$ follows from the fact that $2a_{k+1} < a_k < a_{k+1}/2$. Let now $c = \varrho/(6(\gamma+1))$. By choosing κ and κ' sufficiently large, we have $c/A_1 \geq 2N$. Keeping κ' fixed, and increasing κ further, if necessary, we obtain the first two bounds in (5.3.7) by using that $2^{k+\kappa}a_k \geq 2^{k+\kappa}(k+\kappa')^{-1} \geq c'2^{\kappa}k$, for some constant $c' > 0$. The same inequality, together with $1-\beta_k = a_k/A_k > (k+\kappa')^{-2}2N/c$, implies the third bound in (5.3.7). **QED**

Having verified all of the assumptions made in Section 5.2, we can now apply Theorem 5.2.6 to the n-th step RG transformation $\mathcal{R}_n$, defined by the parameters introduced above. Denote by $\mathcal{L}_n$ the corresponding linear operator from $\mathbb{P}\mathcal{A}_\varrho(\mathcal{V}_{n-1})$ to $\mathbb{P}\mathcal{A}_\varrho(\mathcal{V}_n)$.

Define $\mathcal{A}_{\varrho,k} = \mathcal{A}_\varrho(\mathcal{V}_k)$, for all non-negative integers k. To simplify notation, the norm in $\mathcal{A}_{\varrho,k}$ and the projections $\mathbb{E}$ and $\mathbb{P}$ on this space will not be given indices. From Theorem 5.2.6 we immediately obtain

Theorem 5.3.3 *Let $\gamma \geq 1$. There exist constants $r, C > 0$, such that the following holds, for every positive integer n. Let B_{n-1} be the open ball in $\mathcal{A}_{\varrho,n-1}$ of radius $r\sigma_n^2$, centered at K, where $\sigma_n = \frac{1}{2}(1-\beta_n)\eta_n$. Then $\mathcal{R}_n$ is a bounded analytic map from B_{n-1} to $\mathcal{A}_{\varrho,n}$, satisfying $\|\mathcal{L}_n^{-1}\| \leq 1$ and*

$$
\begin{aligned}
\|(\mathbb{I}-\mathbb{E})\mathcal{R}_n(X)\|_\varrho &\leq C\sigma_n^{-3}\mu_n^{\gamma}\|(\mathbb{I}-\mathbb{E})X\|_\varrho\,,\\
\|(\mathbb{I}-\mathbb{P})\mathcal{R}_n(X)\|_\varrho &\leq C\sigma_n^{-3}\mu_n\|(\mathbb{I}-\mathbb{P})X\|_\varrho\,,\\
\|\mathbb{E}\mathcal{R}_n(X)-\mathcal{R}_n(\mathbb{E}X)\|_\varrho &\leq C\sigma_n^{-6}\mu_n^{-1}\|(\mathbb{I}-\mathbb{E})X\|_\varrho^2\,.
\end{aligned}
\tag{5.3.8}
$$

In what follows, a domain $\mathcal{D}_{n-1}$ for $\mathcal{R}_n$ is a subset of the ball B_{n-1} described in Theorem 5.3.3, that is open in $\mathcal{A}_{\varrho,n-1}$ and contains the vector field K. Given a domain $\mathcal{D}_{n-1}$ for each $\mathcal{R}_n$, the domain $\widetilde{\mathcal{D}}_n$ of the combined RG transformation

$\widetilde{\mathcal{R}}_{n+1} = \mathcal{R}_{n+1} \circ \mathcal{R}_n \circ \ldots \circ \mathcal{R}_1$ is defined recursively as the set of all vector fields in the domain of $\widetilde{\mathcal{R}}_n$ that are mapped under $\widetilde{\mathcal{R}}_n$ into the domain $\mathcal{D}_n$ of $\mathcal{R}_{n+1}$. By Theorem 5.3.3, these domains are open and non-empty, and the transformations $\widetilde{\mathcal{R}}_n$ are analytic.

Theorem 5.3.4 *Let $\gamma \geq 4$. If κ' and then κ are chosen sufficiently large, then there exist a sequence of domains $\mathcal{D}_0, \mathcal{D}_1, \ldots$ for the RG transformations $\mathcal{R}_1, \mathcal{R}_2, \ldots$, such that the set $\mathcal{W} = \cap_n \widetilde{\mathcal{D}}_n$ is the graph of an analytic function $W : (\mathbb{I} - \mathbb{P})\mathcal{D}_0 \to \mathbb{P}\mathcal{D}_0$, satisfying $W(0) = K$ and $DW(0) = 0$. For every $X \in \mathcal{W}$, if $n \geq 1$ and $\psi_n = \prod_{i=1}^n \mu_i$, then*

$$\begin{aligned} \left\|\widetilde{\mathcal{R}}_n(X) - K_n\right\|_\varrho &\leq \psi_n^{1/2} \|(\mathbb{I} - \mathbb{P})X\|_\varrho , \\ \left\|\mathbb{P}[\widetilde{\mathcal{R}}_n(X) - K_n]\right\|_\varrho &\leq \psi_n \|(\mathbb{I} - \mathbb{P})X\|_\varrho^2 , \\ \left\|(\mathbb{I} - \mathbb{E})\widetilde{\mathcal{R}}_n(X)\right\|_\varrho &\leq \psi_n^{\gamma - 1/2} \|(\mathbb{I} - \mathbb{E})X\|_\varrho . \end{aligned} \tag{5.3.9}$$

Proof: Our goal is to apply the stable manifold theorem in Section 4.7. To do so, we first rescale our transformations $\mathcal{R}_n$. Let $r_n = r_{n-1}\sigma_{n+1}^2$ for every positive integer n, with $r_0 > 0$ smaller than half the constant r from Theorem 5.3.3.

Consider the transformations R_n, with $n \in \mathbb{N}$, given by the equation

$$R_n(Z) = r_n^{-1}\left[\mathcal{R}_n(K + r_{n-1}Z) - K\right], \qquad n = 1, 2, \ldots . \tag{5.3.10}$$

The restriction $R_n\mathbb{P}$ defines a linear map from $\mathbb{P}\mathcal{A}_{\varrho,n-1}$ to $\mathbb{P}\mathcal{A}_{\varrho,n}$, which will be denoted by L_n. By Theorem 5.3.3, R_n is analytic and bounded on the ball $\|Z\|_\varrho < 2$, and satisfies

$$
\begin{aligned}
\|(\mathbb{I}-\mathbb{E})R_n(Z)\|_\varrho &\le \varepsilon_n\|(\mathbb{I}-\mathbb{E})Z\|_\varrho\,,\\
\|(\mathbb{I}-\mathbb{P})R_n(Z)\|_\varrho &\le \vartheta_n\|(\mathbb{I}-\mathbb{P})Z\|_\varrho\,,\\
\|\mathbb{P}R_n(Z)-R'_n(\mathbb{P}Z)\|_\varrho &\le \varphi_n\|(\mathbb{I}-\mathbb{E})Z\|_\varrho^2\,,
\end{aligned}
\tag{5.3.11}
$$

where

$$
\varepsilon_n = C\sigma_n^{-3}\sigma_{n+1}^{-2}\mu_n^{\gamma}\,,\quad \vartheta_n = C\sigma_n^{-3}\sigma_{n+1}^{-2}\mu_n\,,\quad \varphi_n = C\sigma_n^{-6}\sigma_{n+1}^{-2}\mu_n^{-1}\,. \tag{5.3.12}
$$

Here, $C \ge 1$ is a constant that may depend on γ, but not on any other RG parameters. In addition, we have $\|L_n^{-1}\| < 1/4$. We will restrict R_n to the domain $D_{n-1} \subset \mathcal{A}_{\varrho,n-1}$, defined by

$$
\|\mathbb{P}Z\|_\varrho < 1\,,\qquad \|(\mathbb{I}-\mathbb{P})Z\|_\varrho < 1\,,\qquad \|(\mathbb{I}-\mathbb{E})Z\|_\varrho < \delta_{n-1}\,, \tag{5.3.13}
$$

where $\delta_{n-1} = (6\varphi_n)^{-1}$. By Proposition 5.3.2, if κ' and κ are chosen sufficiently large, then $C\sigma_n^{-3}\sigma_{n+1}^{-2}\mu_n^{1/2} \le 1/6$ and $C\mu_n^{\gamma-3} \le \sigma_{n+1}^6\sigma_{n+2}^2$, for all positive integers n. These inequalities imply

$$
\varepsilon_n \le \mu_n^{\gamma-1/2}/6\,,\qquad \vartheta_n \le \mu_n^{1/2}/4\,,\qquad \varepsilon_n\delta_{n-1} \le \delta_n\,, \tag{5.3.14}
$$

for all $n \ge 1$. The hypotheses of Theorem 4.7.5 of Section 4.7 are now verified, with $\varepsilon = 1/6$ and $\vartheta = 1/4$, and the conclusions of this theorem imply the statements in Theorem 5.3.4. **QED**

We note that the "min" in equation (5.3.1) could be replaced by "a lower bound", as long as $n \mapsto \Omega'_n$ is a non-increasing sequence of positive real numbers, converging to zero, and the corresponding sequence $\{a_n\}$ is summable. Our estimates are then uniform in the class $\mathcal{B}(\Omega')$ of vectors $\omega \in \mathbb{R}^d$ that admit the same sequence $n \mapsto \Omega'_n$ of lower bounds.

5.4 Construction of invariant tori

Our construction of invariant tori follows closely the ideas from the previous chapters. Consider the RG transformation $\mathcal{R}$ defined in Section 5.2, and a vector field X in the domain of $\mathcal{R}$. If F is any map from D_0 into the domain of $\Lambda_X = \mathcal{U}_X \circ \mathcal{T}_\mu$, define

$$\mathcal{M}_X(F) = \Lambda_X \circ F \circ \mathcal{T}_\mu^{-1}. \tag{5.4.1}$$

Formally, if $\widetilde{\Gamma}$ is an invariant torus for $\mathcal{R}(X)$, then $\Gamma = \mathcal{M}_X(\widetilde{\Gamma})$ is an invariant torus for X. This can be seen easily from the identity $\Lambda_X \circ \Phi^{\eta t}_{\mathcal{R}(X)} = \Phi^t_X \circ \Lambda_X$. In order to make such identities more precise, we estimate the difference between the flow for X and the flow for the constant vector field $K = (\omega, 0)$.

Proposition 5.4.1 *Let τ be a positive real number and X a vector field in $\mathcal{A}_\varrho$, such that $\tau\|X - K\|_\varrho < r < \varrho$. Then for all times t in the interval $[-\tau, \tau]$,*

$$\|\Phi^t_X - \Phi^t_K\|_{\varrho - r} \le \|t(X - K)\|_\varrho. \tag{5.4.2}$$

The proof of this proposition follows standard arguments, using an appropriate integral equation as in Section 4.8, and thus will be omitted.

Consider now a fixed but arbitrary vector field X on the stable manifold $\mathcal{W}$ described in Theorem 5.3.4. Let $X_0 = X$, and $X_n = \mathcal{R}_n(X_{n-1})$ for $n \ge 1$. In order to simplify notation, we will write $\mathcal{U}_k$ and $\mathcal{M}_{k+1}$ in place of $\mathcal{U}_{X_k}$ and $\mathcal{M}_{X_k}$, respectively. Our goal is to construct an appropriate sequence of functions

$\Gamma_k \in \mathcal{A}_0(\mathcal{V}_k)$, satisfying

$$\Gamma_{n-1} = \mathcal{M}_n(\Gamma_n) = \Lambda_n \circ \Gamma_n \circ \mathcal{T}_{\mu_n}^{-1}, \qquad \Lambda_n = \mathcal{U}_{n-1} \circ \mathcal{T}_{\mu_n}, \tag{5.4.3}$$

for all $n > 0$. Then we will show that Γ_0 is an invariant torus for X_0.

For every $n \geq 0$, define $\mathcal{B}_n$ to be the vector space $\mathcal{A}_0(\mathcal{V}_n)$, equipped with the norm

$$\|f\|_n' = r_n^{-1}\|f\|_0 = r_n^{-1} \sum_{v \in \mathcal{V}_n} \|f_v\|, \qquad r_n = \psi_n^{1/3}, \tag{5.4.4}$$

where $\psi_0 = 1$. Denote by B_n the unit ball in $\mathrm{I} + \mathcal{B}_n$, centered at the identity function I, and by $B_n/2$ the ball of radius $1/2$ in the same space.

Proposition 5.4.2 *Let $\gamma \geq 5$. If κ' and then κ are chosen sufficiently large, then there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, such that for every $X \in \mathcal{W} \cap B$, and for every $n \geq 1$, the map $\mathcal{M}_n$ is well defined and analytic, as a function from B_n to $\mathcal{B}_{n-1}$. Furthermore, $\mathcal{M}_n$ takes values in $B_{n-1}/2$, and $\|D\mathcal{M}_n(F)\| \leq \mu_n^{1/4}$, for all $F \in B_n$.*

Proof: Clearly, $\mathcal{M}_n$ is well-defined in some open neighborhood of I in $\mathcal{B}_n$, and

$$\mathcal{M}_n(F) = \mathrm{I} + g + (\mathcal{U}_{n-1} - \mathrm{I}) \circ (\mathrm{I} + g), \qquad g = \mathcal{T}_{\mu_n} \circ f \circ \mathcal{T}_{\mu_n}^{-1}, \tag{5.4.5}$$

where $f = F - \mathrm{I}$. In order to estimate $\mathcal{U}_{n-1} - \mathrm{I}$, we can apply Lemma 5.2.4, with ρ' equal to $\varrho - \varrho(1-\beta_n)/3$, as in the proof of Theorem 5.2.6. We will use Proposition 5.3.2 and assume that κ' and then κ have been chosen sufficiently large, without always mentioning it. By Lemma 5.2.4 and Theorem 5.3.4, there exist a constant $C > 0$, such that

$$\begin{aligned} \|\mathcal{U}_{n-1} - \mathrm{I}\|_{\rho'} &\leq C\sigma_n^{-1}\|\mathbb{I}^- X_{n-1}\|_\varrho \leq C\sigma_n^{-1}\psi_{n-1}^{\gamma-1/2}\|(\mathbb{I}-\mathbb{E})X\|_\varrho \\ &\leq \psi_{n-1}^{\gamma-1}\|(\mathbb{I}-\mathbb{E})X\|_\varrho \leq \psi_n^{3/4}\,, \end{aligned} \tag{5.4.6}$$

for all $n > 1$, and for all $X \in \mathcal{W} \cap B$, provided that the neighborhood B of K has been chosen sufficiently small (depending on κ' and κ). The first inequality in (5.4.6) and the final bound also hold for $n = 1$.

The composition with $\mathrm{I} + g$ in equation (5.4.5) is controlled by Proposition 5.2.1, using that $\|g\|_0 \leq \eta_n^{-1} r_n \|f\|'_n$ is less than $\varrho/2$. Here, and in what follows, we assume that $F \in B_n$. By using that $r_n/r_{n-1} = \mu_n^{1/3}$, we obtain $\|g\|'_{n-1} \leq \eta_n^{-1}\mu_n^{1/3} \leq \mu_n^{2/7}$. When combined with (5.4.6), this shows that $\mathcal{M}_{n-1}$ maps B_n into $B_{n-1}/2$. Using now $\rho' = \varrho/2$, we obtain a bound analogous to (5.4.6) for the derivative of $\mathcal{U}_{n-1}$. This, together with the fact that the inclusion map from B_n into B_{n-1} is bounded in norm by $\mu_n^{1/3}$, shows that $\|D\mathcal{M}_n(F)\| \leq \mu_n^{1/4}$, for all $n \geq 1$, and for all $F \in B_n$. **QED**

Denote by Φ_n and Φ_∞ the flows for the vector fields X_n and K, respectively. In order to prove that a solution to (5.4.3) yields an invariant torus Γ_0 for X, we will use the identity

$$\Phi_{n-1}^t \circ \mathcal{M}_n(F) \circ \Phi_\infty^{-t} = \mathcal{M}_n\big(\Phi_n^{\eta_n t} \circ F \circ \Phi_\infty^{-\eta_n t}\big)\,, \tag{5.4.7}$$

which follows from the relation described after (5.4.1), between the flow for a vector field and the flow for the corresponding renormalized vector field. This requires an estimate of the following type.

Proposition 5.4.3 *Under the same assumptions as in Proposition 5.4.2, there* *exists an open neighborhood B of K in $\mathcal{A}_\varrho$, such that for every $X \in \mathcal{W} \cap B$, and for every $n \geq 1$, the function $\Phi_n^s \circ F \circ \Phi_\infty^{-s}$ belongs to B_n, whenever $F \in B_n/2$ and $|s| \leq \psi_n^{-1/6}$.*

Proof: We will use the identity

$$\Phi_n^s \circ F \circ \Phi_\infty^{-s} = \mathrm{I} + f \circ \Phi_\infty^{-s} + \left[\Phi_n^s \circ \Phi_\infty^{-s} - \mathrm{I}\right] \circ \left(\mathrm{I} + f \circ \Phi_\infty^{-s}\right). \tag{5.4.8}$$

By Proposition 5.4.1 and Theorem 5.3.4, we have the bound

$$\left\|\Phi_n^s \circ \Phi_\infty^{-s} - \mathrm{I}\right\|_{\varrho/2} \leq \|s(X_n - K)\|_\rho \leq C\psi_n^{1/3}\|(\mathbb{I} - \mathbb{P})X\|_\varrho, \tag{5.4.9}$$

provided e.g. that the right hand side of this inequality is less than $\varrho/2$. This is certainly the case, for any n, if $\|X - K\|_\varrho$ is sufficiently small. The composition by $\mathrm{I} + f \circ \Phi_\infty^{-s}$ in equation (5.4.8) is controlled in the same way as the composition by $\mathrm{I} + g$ in the proof of Proposition 5.4.2, using also that $\|f \circ \Phi_\infty^{-s}\|_0 = \|f\|_0$. As a result, the third term on the right hand side of (5.4.8) belongs to $\mathcal{B}_n$ and is bounded in norm by $C\|X - K\|_\varrho$, which is less than $1/2$ for any $n \geq 1$, if X is sufficiently close to K. **QED**

Now we are ready to construct invariant tori. A function f defined on $\mathcal{W}$ is said to be analytic if $f \circ W$ is analytic on the domain of W.

Theorem 5.4.4 *Under the same assumptions as in Proposition 5.4.2, there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, such that the following holds. Given any $X \in \mathcal{W} \cap B$, and any sequence of functions $F_k \in B_k$, define*

$$\Gamma_{n,k} = \left(\mathcal{M}_{n+1} \circ \ldots \circ \mathcal{M}_k\right)(F_k), \qquad 0 \leq n < k. \tag{5.4.10}$$

Then the limits $\Gamma_n = \lim_{k\to\infty} \Gamma_{n,k}$ exist in $\mathcal{B}_n$, are independent of the choice of $F_0, F_1, \ldots$, and satisfy the identities (5.4.3). *Furthermore, Γ_0 is an elliptic invariant torus for X, and the map $X \mapsto \Gamma_0$ is analytic and bounded on $\mathcal{W} \cap B$.*

Proof: By Proposition 5.4.2 and Proposition 5.3.2, the map $\mathcal{M}_n : B_n \to B_{n-1}/2$ contracts distances by a factor of at least $1/2$. Thus, if $1 \le n < k < k'$, then the difference $\Gamma_{n,k'} - \Gamma_{n,k}$ is bounded in norm by 2^{n-k+1}. This shows that the sequence $k \mapsto \Gamma_{n,k}$ converges in $\mathcal{B}_n$ to a limit Γ_n, which is independent of the choice of the functions F_k. By choosing $F_k = \Gamma_k$ for all k, we obtain the identities (5.4.3). The analyticity of $X \mapsto \Gamma_0$ follows via chain rule from the analyticity of the maps used in our construction, and from uniform convergence.

In order to prove that Γ_0 is an invariant torus for X, we will use the identity (5.4.7). To be more precise, given a real number $-1 < t < 1$, define $t_n = \lambda_n t$ for all $n \ge 0$. By using that $\lambda_n \le \psi_n^{-1/6}$, independently of n, if κ' and κ have been chosen sufficiently large (which we assume), Proposition 5.4.3 allows us to iterate the identity (5.4.7), and get the identity

$$\Phi_0^t \circ \Gamma_{0,k} \circ \Phi_\infty^{-t} = \bigl(\mathcal{M}_1 \circ \ldots \circ \mathcal{M}_k\bigr)\bigl(\Phi_k^{t_k} \circ \Phi_\infty^{-t_k}\bigr), \tag{5.4.11}$$

for all $k > 0$. As proved above, the right (and thus left) hand side of this equation converges in $\mathcal{A}_0$ to Γ_0. In addition, $\Gamma_{0,k} \to \Gamma_0$ in $\mathcal{A}_0$, and the convergence is pointwise as well, by part (i) of Proposition 5.2.1. Thus, since the flow Φ_0^t is continuous, we have $\Phi_0^t \circ \Gamma_0 \circ \Phi_\infty^{-t} = \Gamma_0$. This identity now extends to arbitrary $t \in \mathbb{R}$, due to the group property of the flow, and the fact that composition with Φ_∞^s is an isometry on $\mathcal{A}_0$.

Finally, notice that $\lambda_n \|DX_n\|_{\varrho/2}$ is an upper bound on the modulus of the Lyapunov exponent for the flow of $\lambda_n X_n$ on the range of Γ_n. Since X_0 is obtained from $\lambda_n X_n$ by a change of variables, and Γ_0 is the corresponding invariant torus for X_0, the same upper bound applies to the flow for X_0 on the torus Γ_0. But by Theorem 5.3.4, $\lambda_n \|DX_n\|_{\varrho/2} \to 0$ as $n \to \infty$. This shows that the torus Γ_0 is elliptic. **QED**

In what follows, the torus Γ_0 associated with a vector field $X \in \mathcal{W}$ will be denoted by Γ_X. For convenience, we extend the map $X \mapsto \Gamma_X$ to an open neighborhood of K, by setting $\Gamma_X = \Gamma_{X'}$, where $X' = (\mathbb{I} + W)(X - \mathbb{P}X)$.

Theorem 5.4.5 *Let $\rho > \varrho + \delta$ with $\delta > 0$. Under the same assumptions as in Proposition 5.4.2, there exists an open neighborhood B of K in $\mathcal{A}_\rho(\mathcal{V}_0)$, such that Γ_X has an analytic continuation to $\|\mathrm{Im}\, x\| < \delta$, for each $X \in B$. With this continuation, $X \mapsto \Gamma_X$ defines a bounded analytic map from B to $\mathcal{A}^0_\delta(\mathcal{V}_0)$.*

A proof of this theorem is completely analogous to the proof of Theorem 4.8.5 in the previous chapter. For that reason, we will just give a sketch here.

Consider the translations $R_u(x, y) = (x + u, y)$. By examining the construction of $\mathcal{W}$ and Γ_X, one verifies that for any $u \in \mathbb{R}^d$, the translated vector field $R_u^* X$ belongs to $\mathcal{W}$ whenever X does, and that

$$\Gamma_X(u, 0) = (R_u \circ \Gamma_{R_u^* X})(0, 0)\,. \tag{5.4.12}$$

The idea now is to use the analyticity of map $X \mapsto \Gamma_X$, to extend the right hand side of equation (5.4.12) to the complex domain $\|\mathrm{Im}\, u\| < \delta$. This yields the desired analytic continuation of Γ_X. The remaining parts of Theorem 5.4.5 are proved by using that the right hand side of identity (5.4.12) is jointly analytic in X and u.

This theorem, together with Theorem 5.3.4, implies Theorem 5.1.1.

Bibliography

[1] J.J. Abad and H. Koch, Renormalization and periodic orbits for Hamiltonian flows, *Commun. Math. Phys.* **212** (2000), 371-394.

[2] J.J. Abad, H. Koch, and P. Wittwer, A Renormalization Group for Hamiltonians: Numerical Results, *Nonlinearity* **11** (1998), 1185-1194.

[3] V.I. Arnold, Proof of A. N. Kolmogorov's theorem on the preservation of quasiperiodic motions under small perturbations of the Hamiltonian, *Usp. Mat. Nauk. SSSR* **18**(5) (1963), 13-40.

[4] L. Bernstein, The Jacobi-Perron algorithm, its theory and application, *Lecture Notes in Mathematics* **207**, Springer (Berlin-Heidelberg-New York, 1971).

[5] A. Berretti and G. Gentile, Bryuno function and the standard map, *Commun. Math. Phys.* **220** (2001), 623-656.

[6] J. Bricmont, K. Gawedzki and A. Kupiainen, KAM theorem and quantum field theory, *Commun. Math. Phys.* **201** 3 (1999), 699-727.

[7] A.D. Brjuno, Analytic form of differential equations I, *Trudy Moskov. Mat. Obshch.* **25** (1971), 119-262; *Trans. Moscow Math. Soc.* **25** (1973), 131-288.

[8] A.D. Brjuno, Analytic form of differential equations II, *Trudy Moskov. Mat. Obshch.* **26** (1972), 199-239; *Trans. Moscow Math. Soc.* **26** (1974), 199-239.

[9] H.W. Broer, KAM theory: The legacy of Kolmogorov's 1954 paper, *Bull. Amer. Math. Soc.* **41** (2004), 507-521.

[10] H.W. Broer, G.B. Huitema, and M.B. Sevryuk, Quasi-periodicity in families of dynamical systems: Order amidst chaos, in *Lecture Notes in Mathematics* **1645**, Springer Verlag (Berlin, 1996).

[11] J.W.S. Cassels, An introduction to Diophantine approximation, Cambridge University Press (Cambridge, 1957).

[12] C. Chandre, H.R. Jauslin, G. Benfatto and A. Cellleti, An approximate renormalization-group transformation for Hamiltonian systems with three degrees of freedom, *Phys. Rev. E* **60** (1999), 5412-5421.

[13] C.E. Delaunay, Mémoire sur une nouvelle méthode pour la détermination du mouvement de la Lune, *C. R. Acad. Sci.* **22** (1846), 32.

[14] C.E. Delaunay, Théorie du Mouvement de la Lune, 2 Vols. in *Mem. Acad. Sci.* **28** and **29** (Mallet-Bachelier, Paris, 1860 and Gauthier-Villars, Paris, 1867).

[15] C. Chandre and P. Moussa, Scaling law for the critical function of an approximate renormalization, *Nonlinearity* **14** (2001), 803-816.

[16] C. Chandre and H.R. Jauslin, Renormalization-group analysis for the transition to chaos in Hamiltonian systems, *Physics Reports* **365** (2002), 1-64.

[17] P. Coullet and C. Tresser, Iteration d'endomorphismes et groupe de renormalisation, *J. Phys. Colloque* **39** (1978), C5-25.

[18] S.G. Dani, Divergent trajectories of flows on homogeneous spaces and Diophantine approximation, *J. Reine Angew. Math* **359** (1985), 55-89.

[19] J. Ecalle and B. Valet, Bruno correction and linearization of resonant vector fields and diffeomorphisms, *Math. Z.* **229** (1998), 249-318.

[20] L.H. Eliasson, Absolutely Convergent Series Expansions for Quasi Periodic Motions, Report 2-88, Dept. of Mathematics, University of Stockholm (1988); *Math. Phys. Electron. J.* **2**(4) (1996), 33.

[21] A. Epstein and M. Yampolsky, A universal parabolic map, *Ergod. Th. & Dynam. Sys.*, to appear.

[22] D.F. Escande and F. Doveil, Renormalization method for computing the threshold of the large-scale stochastic instability in two degree-of-fredom Hamiltonian systems, *J. Stat. Phys.* **26** (1981), 257-284.

[23] M.J. Feigenbaum, Quantitative universality for a class of nonlinear transformations, *J. Stat. Phys.* **19** (1978), 25-52.

[24] M.J. Feigenbaum, The universal metric properties of nonlinear transformations, *J. Stat. Phys.* **21** (1979), 669.

[25] M.J. Feigenbaum, L.P. Kadanoff and S.J. Shenker, Quasiperiodicity in dissipative systems: A renormalization analysis, *Physica* **5D** (1982), 370-386.

[26] D. Gaidashev, Renormalization of isoenergetically degenerate Hamiltonian flows and associated bifurcations of invariant tori, *Discrete & Contin. Dynam. Sys.* **13** (2005), 63-102.

[27] D. Gaidashev and H. Koch, Renormalization and shearless invariant tori: Numerical results, *Nonlinearity* **17** (2004), 1713-1722.

[28] G. Gallavotti, G. Gentile, and V. Mastropietro, Field theory and KAM tori, *Math. Phys. Electron J.* **1**(5) (1995), 13.

[29] G. Gallavotti and G. Gentile, Hyperbolic low-dimensional invariant tori and summations of divergent series, *Commun. Math. Phys.* **227** (2002), 421-460.

[30] G. Gentile and V. Mastropietro, Methods for the analysis of the Lindstedt series

for KAM tori and renormalizability in classical mechanics (A review with some applications), *Rev. Math. Phys.* **8** (1996), 393-444.

[31] G. Gentile, M.V. Bartuccelli and J.H.B. Deane, Summation of divergent series and Borel summability for strongly dissipative differential equations with periodic or quasiperiodic forcing terms, *J. Math. Phys.* **46** (2005), 062704.

[32] G. Gentile, M.V. Bartuccelli and J.H.B. Deane, Quasi-periodic attractors, Borel summability and the Bryuno condition for strongly dissipative systems, *J. Math. Phys.* **47** (2006), 072702.

[33] M. Govin, C. Chandre and H.R. Jauslin, Kolmogorov-Arnol'd-Moser-Renormalization group analysis of stability of Hamiltonian flows, *Phys. Rev. Lett.* **79** (1997), 3881.

[34] G.H. Hardy and E.M. Wright, An introduction to the theory of numbers, Oxford University Press (New York, 1979).

[35] N.T.A. Haydn, On invariant curves under renormalization, *Nonlinearity* **3** (1990), 887-912.

[36] R.D. Hazeltine and J.D. Meiss, *Plasma Confinement*, Addison-Wesley (Redwood City, CA, 1991).

[37] M.R. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, *Publications mathématiques de l'I.H.É.S.*, **49** (1979), 5-233.

[38] L.P. Kadanoff, Scaling laws for Ising models near T_c, *Physics* **2** (1966), 263.

[39] L.P. Kadanoff, Scaling for a critical Kolmogorov-Arnold-Moser trajectory, *Phys. Rev. Lett.* **47** (1981), 1641-1643.

[40] K.M. Khanin and Ya.G. Sinai, The renormalization group method and Kolmogorov-Arnol'd-Moser theory, in *Nonlinear Phenomena in Plasma*

Physics and Hydrodynamics, edited by R. Z. Sagdeev, Mir (Moscow, 1986), 93-118.

[41] K. Khanin, J. Lopes-Dias and J. Marklof, Multidimensional continued fractions, dynamical renormalization and KAM theory, *Comm. Math. Phys* **270** (2007), 197-231.

[42] K. Khanin, J. Lopes-Dias and J. Marklof, Renormalization for multidimensional Hamiltonian flows, *Nonlinearity* **19** (2006), 2727.

[43] D.Y. Kleinbock and G.A. Margulis, Flows on homogeneous spaces and Diophantine approximation on manifolds, *Ann. of Math. (2)* **148** (1998), 339-360.

[44] H. Koch, A renormalization group for Hamiltonians, with applications to KAM tori, *Ergod. Th. & Dynam. Sys.* **19** (1999), 475-521.

[45] H. Koch, On the renormalization of Hamiltonian flows, and critical invariant tori, *Discrete & Contin. Dynam. Sys.* **8** (2002), 633-646.

[46] H. Koch, A Renormalization Group Fixed Point Associated with the Breakup of Golden Invariant Tori, *Discrete & Contin. Dynam. Sys.* **11** (2004), 881-909.

[47] H. Koch, Existence of Critical Invariant Tori, *Erg. Theor. Dyn. Syst.*, to appear (2006).

[48] H. Koch and J. Lopes Dias, Renormalization of Diophantine skew flows, with applications to the reducibility problem, preprint: mp_arc 05-285 (2005).

[49] S. Kocić, S. Mahajan and R.D. Hazeltine, Topology of plasma equilibria and the current closure condition, *Phys. Rev. E* **71** (2005), 057401.

[50] S. Kocić, Renormalization of Hamiltonians for Diophantine frequency vectors and KAM tori, *Nonlinearity* **18** (2005), 2513-2544.

[51] A.N. Kolmogorov, On conservation of quasiperiodic motions under small perturbations of the Hamiltonian, *Dokl. Akad. Nauk. SSSR* **98** (1954), 527-530.

[52] S. Kovalevskaya, Zur Theorie der partiellen Differentialgleichungen, Dissertation (Göttingen, 1874); *Journal fur die reine und angewandte Mathematik* **80** (1875), 1-32.

[53] J.C. Lagarias, Geodesic multidimensional continued fractions, *Proc. London Math. Soc.* **69** (1994), 464-488.

[54] O.E. Lanford III, Renormalization group methods for circle mappings, in *Nonlinear evolution and chaotic phenomena*, edited by G. Gallavotti and P.F. Zweifel, Plenum Press (New York, 1988), 25-36.

[55] R. de la Llave, A tutorial on KAM theory, in *Proceedings of Symposia in Pure Mathematics* **69**, edited by A. Katok et al, Amer. Math. Soc. (Providence, 2001), 175-292.

[56] J. Lopes-Dias, Renormalization of flows on the multidimensional torus close to a KT frequency vector, *Nonlinearity* **15** (2002), 647-664.

[57] J. Lopes-Dias, Renormalization scheme for vector fields on $\mathbb{T}^2$ with a diophantine frequency, *Nonlinearity* **15** (2002), 665-679.

[58] J. Lopes-Dias, Brjuno condition and renormalization for Poincaré flows, *Discrete & Contin. Dynam. Sys.* **15** (2006), 641-656.

[59] J. Lopes-Dias, A normal form theorem for Brjuno skew-systems through renormalization, *J. Differential Equations* **230** (2006), 1-23.

[60] J. Lopes-Dias, Differentiable strengthening of conjugacies for analytic torus flows, *J. Differential Equations*, to appear (2006).

[61] R.S. MacKay, Renormalization in area-preserving maps, Ph.D. Thesis, Princeton University (Princeton, 1982).

[62] R.S. MacKay, A renormalization approach to invariant circles in area-preserving maps, *Physica D* **7** (1983), 283-300.

[63] R.S. MacKay, Exact results for an approximate renormalization scheme and some predictions for the break-up of invariant tori, *Physica D* **33** (1988), 240-265.

[64] R.S. MacKay, Three topics in Hamiltonian dynamics, in *Proceedings of the International Conference on Dynamical Systems and Chaos* **2**, edited by Y. Aizawa, S. Saito and K. Shiraiwa, World Scientific (Singapore, 1995), 34-43.

[65] R.S. MacKay, J. D. Meiss and J. Stark, An approximate renormalization for the break-up of invariant tori with three frequencies, *Physics Letters A* **190** (1994), 417-424.

[66] J. Moser, On invariant curves of area-preserving mappings of an annulus, *Nachr. Akad. Wiss. Gött. II, Math. Phys.* **K1** (1962), 1-20.

[67] J. Moser, Convergent series expensions for quasi-periodic motions, *Math. Ann.* **169** (1967), 136-176.

[68] S. Ostlund, D. Rand, J. Sethna and E. Siggia, Universal properties of the transition from quasi-periodicity to chaos in dissipative systems, *Physica D* **8** (1983), 303-342.

[69] J. Pöschel, Integrability of Hamiltonian systems on Cantor sets, *Comm. Pure Appl. Math.* **35** (1989), 653-696.

[70] H. Poincaré, *Méthodes Nouvelles de la Mécanique Céleste* (Gauthier-Villars et fils, Paris, 1892-99); *New Methods of Celestial Mechanics*, edited and introduced by Daniel L. Goroff, American Institute of Physics (Woodbury, NY, 1993).

[71] H. Rüssmann, On the frequencies of quasi-periodic solutions of analytic nearly integrable Hamiltonian systems, in *Seminar on Dynamical Systems*, Euler

Int. Math. Inst. (St. Petersburg, 1991); PNLDE **12**, edited by S. Kuksin, V. Lazutkin, J. Pöschel, Birkhäuser Verlag (Basel, 1994) 160-183.

[72] H. Rüssmann, Invariant tori in non-degenerate nearly integrable Hamiltonian systems, *Regul. Chaotic Dyn.* **6** (2001), 119-204.

[73] M.B. Sevryuk, *Reversible systems*, Lecture Notes in Mathematics **1211**, Springer Verlag (Berlin, 1986).

[74] S.J. Shenker and L.P. Kadanoff, Critical behavior of a KAM surface: Empirical results, *J. Stat. Phys.* **27**(4) (1982), 631-656.

[75] A. Stirnemann, Renormalization for golden circles, *Comm. Math. Phys.* **152** (1993), 369-431.

[76] E.C.G. Stueckelberg et A. Petermann, La normalisation des constantes dans la theorie des quanta, *Helv. Phys. Acta* **26** (1953), 499-520.

[77] J.C. Yoccoz, Petits diviseurs en dimension 1, *Astérisque* **231** (1995).

[78] J.C. Yoccoz, Analytic linearization of circle diffeomorphisms, *Dynamical systems and small divisors*, Lecture Notes in Mathematics **1784**, edited by Marmi and Yoccoz, Springer-Verlag (New York, 2002).

Vita

Saša Kocić was born in Jagodina, Serbia, on October 19, 1973, the son of Jagoda Kocić and Mića Kocić. After completing his work at Jagodina's Grammer School and one year of mandatory military service in 1993, he entered The University of Belgrade in Belgrade, Serbia. He received the degree of Bachelor of Science in Physics in June 1998 and Master of Science in Physics (Group: Theoretical Condensed Matter Physics) in July 2000 from The University of Belgrade. In August 2000, he enrolled in the Graduate School of The University of Texas at Austin.

Permanent Address: 5106 N Lamar Blvd 221, Austin, Texas 78751

This dissertation was typeset with LaTeX 2_ε[1] by the author.

[1] LaTeX 2_ε is an extension of LaTeX. LaTeX is a collection of macros for TeX. TeX is a trademark of the American Mathematical Society. The macros used in formatting this dissertation were written by Dinesh Das, Department of Computer Sciences, The University of Texas at Austin, and extended by Bert Kay, James A. Bednar, and Ayman El-Khashab.